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BULLETIN OF THE
AMERICAN
MATHEMATICAL SOCIETY.

CONTINUATION OF THE BULLETIN OF THE NEW YORK
MATHEMATICAL SOCIETY.

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

F. N. COLE A. ZIWET D. E. SMITH F. MORLEY

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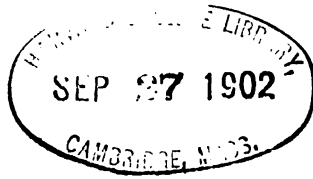
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BULLETIN OF THE
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SOME INSTRUCTIVE EXAMPLES IN THE
CALCULUS OF VARIATIONS.

BY PROFESSOR OSKAR BOLZA.

(Read before the American Mathematical Society, September 2, 1902.)

In the following note I propose to give some examples which illustrate in a simple manner several points of fundamental importance in the calculus of variations.

§ 1. *The General Problem and its Assumptions.*

We consider the problem * to minimize the integral

$$I = \int_{x_0}^{x_1} F(x, y, y') dx \quad (1)$$

under the following *assumptions*:

1. The function $F(x, y, z)$ considered as a function of the three independent variables x, y, z is real and regular † in the vicinity of every finite real point $x = a, y = b, z = c$ for which $x = a, y = b$ lies in a given region R of the xy -plane.

2. The functions $y = f(x)$ admitted to consideration satisfy the following conditions:

(a) For the given end values $x = x_0, x = x_1, y$ takes the given values $y = y_0$ and $y = y_1$ respectively, i. e., the "curves" $y = f(x)$ pass through two given points $A: (x_0, y_0)$ and $B: (x_1, y_1)$;

* Compare for the formulation of the problem Osgood's article "Sufficient conditions in the calculus of variations," *Annals of Math.*, 2nd ser., vol. 2 (1901), p. 105. We deviate, however, in several points from Osgood's assumptions.

† i. e. developable into an ordinary power series in $x - a, y - b, z - c$.

(b) y is continuous on the whole interval (x_0x_1) , and the first derivative y' exists and is continuous on (x_0x_1) with the possible exception of a finite number of points; in these exceptional points the progressive and regressive derivatives exist and are finite;

(c) The "curves" $y = f(x)$ lie, for $x_0 \leq x \leq x_1$ in the interior of the region R .

The totality of functions (curves) satisfying these conditions will be denoted by $\mathfrak{M}$.

A function $y = f(x)$ of $\mathfrak{M}$ is said to *minimize* the integral I if there exists a positive quantity ϵ such that the total variation

$$\Delta I = \int_{x_0}^{x_1} F(x, \bar{y}, \bar{y}') dx - \int_{x_0}^{x_1} F(x, y, y') dx$$

is positive for every function $\bar{y} = \bar{f}(x)$ of $\mathfrak{M}$, different from y , for which

$$|\Delta y| < \epsilon,$$

where

$$\Delta y = \bar{y} - y.$$

It is well known that the following conditions are *necessary* for a minimum:

1. The minimizing function $y = f(x)$ must be an *extremal*, i. e., satisfy *Euler's differential equation*:*

$$F_y - \frac{d}{dx} F_{y'} = 0, \quad (\text{I})$$

literal subscripts denoting partial derivatives.

2. *Legendre's condition*:

$$F_{y'y'}(x, f(x), f'(x)) \geq 0 \text{ on } (x_0x_1). \quad (\text{II})$$

3. *Jacobi's condition*:†

$$x_1 \leq x'_0 \quad (\text{III})$$

where x'_0 denotes the "conjugate" of x_0 , x_0 being supposed less than x_1 .

* See for instance Whittemore, *Annals of Math.*, 2nd ser., vol. 2 (1901), p. 130.

† See Erdmann, *Schlömilch's Zeitschrift*, vol. 23 (1878), p. 367.

4. *Weierstrass's condition* : *

$$\frac{E(x, f(x), f'(x), p)}{(p - f'(x))^2} \geq 0 \quad (\text{IV})$$

on (x_0, x_1) and for every finite value of p , the function

$$E(x, y, y', p)$$

being defined by

$$\begin{aligned} E(x, y, y', p) &= F(x, y, p) - F(x, y, y') \\ &\quad - (p - y') F_{y'}(x, y, y'). \end{aligned} \quad (2)$$

Since by Taylor's theorem

$$E(x, y, y', p) = \frac{(p - y')^2}{2} F_{y'y'}(x, y, y' + \theta(p - y')) \quad (3)$$

where $0 < \theta < 1$, it follows that (IV) is always satisfied whenever

$$F_{y'y'}(x, f(x), p) \geq 0 \quad (\text{II}_a)$$

on (x_0, x_1) for every finite value of p .

By (II'), (II_a'), (III'), (IV'), we denote the inequalities (II), (II_a), (III), (IV) after the suppression of the equality sign.

§ 2. *Example Illustrating the Necessity of Weierstrass's Condition.*

After Jacobi had discovered condition (III), it was generally believed that conditions (I), (II'), (III') were also *sufficient* for a minimum, until Weierstrass proved in 1879 their insufficiency and added his fourth necessary condition.

The following example illustrates this point :

Example I: To minimize the integral

$$I = \int_{x_0}^{x_1} y'^2 (y' + 1)^2 dx. \quad (4)$$

* Weierstrass, *Lectures*, 1879; compare also Osgood, *l. c.*, p. 118, and Hilbert, *Archiv für Mathematik* (3), vol. 1 (1901), p. 231. The expansion of E according to powers of $p - y'$ shows that the quotient $E(x, y, y', p) / (p - y')^2$ remains regular even when $p = y'$.

Here the extremals are straight lines; let

$$\mathfrak{C}: \quad y = mx + n$$

be the extremal through the two given points $A: (x_0, y_0)$ and $B: (x_1, y_1)$.

Further, we have

$$F_{yy'} = 2(6y'^2 + 6y' + 1).$$

Hence

$$F_{yy'}(x, f(x), f'(x)) = 2(6m^2 + 6m + 1).$$

The roots of the equation

$$6m^2 + 6m + 1 = 0$$

are

$$m_1 = \frac{1}{2}(-1 + 1/\sqrt{3}) = -0.2113 \dots$$

$$m_2 = \frac{1}{2}(-1 - 1/\sqrt{3}) = -0.7887 \dots$$

Hence, if we suppose the two given points A and B so situated that

$$\text{either } m > m_1 \text{ or } m < m_2,$$

condition (II') will be satisfied.

Further, condition (III') is always satisfied; for since the extremals through the point A are straight lines, it follows that there exists no conjugate point on $\mathfrak{C}$, however far we may go.

Finally

$$\frac{E(x, f(x), f'(x), p)}{(p - f'(x))^2} = p^2 + 2(m+1)p + 3m^2 + 4m + 1;$$

it is positive for every p when $m(m+1) > 0$; it can change sign when $m(m+1) < 0$.

Accordingly we have to distinguish the following three cases:

Case I: $-1 < m < 0$.

In this case the line AB does not minimize the integral I . This can be seen at once as follows: However small we may select a positive quantity ϵ , we can always draw in the "neighborhood (ϵ)" * of AB a broken line $\bar{\mathfrak{C}}$ joining A and B and

* Compare, for the definition of neighborhood, Osgood, *l. c.*, p. 106.

made up of segments having alternately the slope 0 and -1 . For such a broken line the integral $\bar{I}$ is evidently equal to

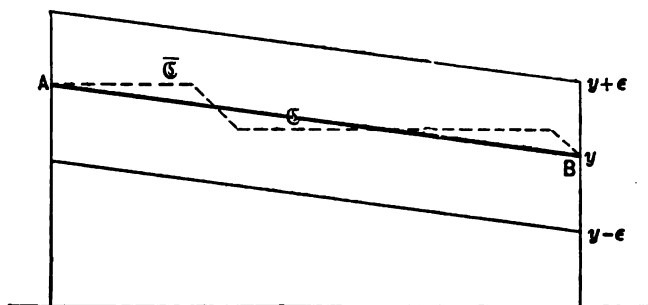


FIG. 1.

zero, the integrand being zero between the limits of integration, whereas for the line AB the integral I has the positive value

$$I_0 = m^2(m+1)^2 \cdot (x_1 - x_0).$$

Case II: $m = 0$ or $m = -1$.

In this case the line AB minimizes the integral I ; for it furnishes the value $I = 0$ whereas every other curve joining A and B and representable in the form $y = f(x)$ furnishes a positive value.

Case III: $m > 0$ or $m < -1$.

In this case the line AB minimizes the integral I . For the set of straight lines parallel to AB constitutes a "field of extremals" about AB , hence by Weierstrass's* theorem we have for every curve $\bar{\epsilon}$ not coinciding with AB , joining A and B and satisfying the conditions of § 1,

$$\Delta I = \int_{x_0}^{x_1} E(x, y, y', \bar{y}') dx, \quad (5)$$

where x, y is a point of $\bar{\epsilon}$; $\bar{y}'$ the slope of $\bar{\epsilon}$ at x, y ; y' the slope at x, y of the extremal of the field passing through x, y , i. e., $y' = m$; therefore

$$E(x, y, y', \bar{y}') = (\bar{y}' - m)^2 [\bar{y}'^2 + 2(m+1)\bar{y}' + 3m^2 + 4m + 1],$$

* Compare Osgood, l. c., p. 118; also Hilbert, *Archiv für Mathematik* (3) vol. 1 (1901), p. 231.

which is positive except where $\bar{y}' = m$. Hence it follows that

$$\Delta I > 0.$$

Our example illustrates in this case another point, viz., that condition (II_a) is not necessary for a minimum. Indeed

$$F_{y'y'}(x, f(x), p) = 2(6p^2 + 6p + 1),$$

which may take negative as well as positive values.

§ 3. *Relation between a Problem in Parameter Representation and the Corresponding Problem with x as Independent Variable.*

When one has become acquainted with Weierstrass's treatment of the simplest type of problems of the calculus of variations (viz., in parameter representation) one is inclined to regard the older method—in which x is taken as the independent variable—as antiquated and imperfect as compared with Weierstrass's method; unjustly however, for *the two methods deal with two clearly distinct questions* and which of the two deserves the preference depends upon the nature of the special problem under consideration.

When it is proposed to minimize the integral

$$I = \int_{t_0}^{t_1} F\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right) dt, \quad (6)$$

where

$$F(x, y, kx', ky') = k F(x, y, x', y') \quad (7)$$

for every positive k , we consider a certain well-defined manifoldness $\mathfrak{N}$ of curves

$$x = \phi(t), \quad y = \psi(t),$$

out of which we have to select those curves which minimize the integral I .

If we consider, instead of $\mathfrak{N}$, another manifoldness $\mathfrak{M}$ of curves, we obtain an entirely different problem.

Let now $\mathfrak{M}$ be, in particular, the totality of those curves of $\mathfrak{N}$ which are representable in the form

$$y = f(x),$$

$f(x)$ being a single valued function of x . Then the new problem may be formulated thus: Among all curves (*functions*) $y=f(x)$ of the totality $\mathfrak{M}$ to find those which minimize the integral

$$I = \int_{x_0}^{x_1} F\left(x, y, 1, \frac{dy}{dx}\right) dx. \quad (8)$$

We distinguish the two problems as “the problem (T)” and “the problem (X).” Since $\mathfrak{M}$ is contained in $\mathfrak{N}$, it follows that *every solution of (T) which is representable in the form $y=f(x)$ is a fortiori also a solution of (X); but (T) may have solutions which are not representable in this form and which accordingly are not solutions of (X).*

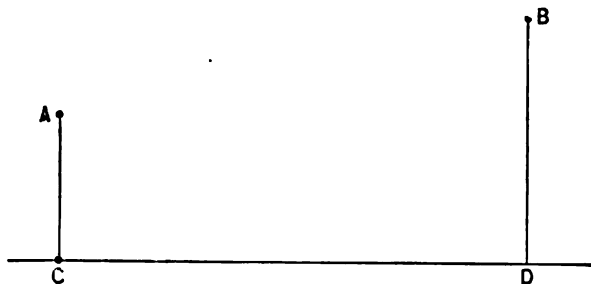


FIG. 2.

A well-known example of this kind is the so-called “discontinuous solution” of the problem of the minimum surface of revolution,* which consists of two perpendiculars AC and DB to the axis of revolution (the x -axis) and the segment CD of the latter. The broken line $ACDB$ is under certain assumptions concerning the positions of A and B a solution of the problem (T), but not of the problem (X), because the segments AC and BD are not representable in the form $y=f(x)$.

On the other hand, *the problem (X) may have solutions which are not at the same time solutions of (T).*

An example of this kind is furnished by the problem of § 2, case III. In this case the line AB furnishes, as we have seen, a smaller value for the integral

* See Kneser, *Lehrbuch der Variationsrechnung*, p. 179.

$$I = \int_{x_0}^{x_1} y'^2 \left(+ y'1 \right)^2 dx,$$

than any other curve joining A and B , and representable in the form $y = f(x)$. But if we pass to the corresponding problem (T') and admit also curves which are not representable in this form, we can easily find neighboring curves which furnish still smaller values for the integral I than the line AB ; for instance any broken line $\bar{C}$ joining A and B and made up of

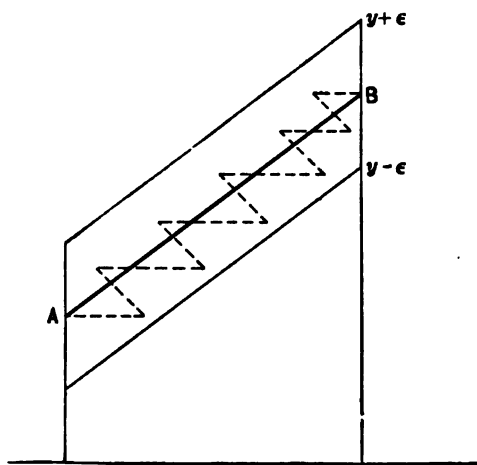


FIG. 3.

segments having alternately the slope 0 and -1 . For such a broken line the integral is zero, whereas it is positive for the line AB . Hence AB is a solution of (X) but not of (T').

§ 4. *Example Illustrating the Insufficiency of Weierstrass's Condition in Case x is Taken as Independent Variable.*

Weierstrass discusses the problem to minimize the integral (6) under the assumption that $F(x, y, x', y')$, considered as a function of its four arguments is regular in the vicinity of every point $x = a, y = b, x' = a', y' = b'$ for which a, b lies in a certain region of the xy -plane, while a', b' may have any system of values except $a' = 0, b' = 0$. Under this assumption he proves that the conditions corresponding to (I), (II'), (III'), (IV') are *sufficient* for a minimum.

Nevertheless, for a minimum of the integral (1) under the assumptions stated in § 1, the conditions (I), (II'), (III'), and (IV') are not sufficient, not even if (IV') be replaced by the stronger condition (II')*.

This is shown by the following example:

Example II.—To minimize the integral

$$I = \int_0^1 (ay'^2 - 4byy'^3 + 2bxy'^4) dx, \quad (9)$$

a, b being two positive constants, with the initial conditions

$$y = 0 \text{ for } x = 0 \quad \text{and} \quad y = 0 \text{ for } x = 1.$$

Here Euler's equation reduces to

$$-y''F_{y'y'} = 0, \quad (I)$$

where $F_{y'y'} = 2a - 24byy' + 24bxy'^2$.

The only extremal through the two given points $A: (0, 0)$ and $B: (1, 0)$ is the straight line

$$\mathfrak{C}: \quad y = 0.$$

Further

$$F_{y'y'}(x, f(x), f'(x)) = 2a > 0. \quad (II')$$

The set of extremals through A is the pencil of straight lines through A ; hence there exists no conjugate point and condition (III') is satisfied.

Further

$$\frac{E(x, y, y', p)}{(p - y')^2} = (a - 8byy' + 6bxy'^2)$$

$$+ p(-4by + 4bxy') + 2bxp^2.$$

Hence along $\mathfrak{C}$

$$\frac{E(x, f(x), f'(x), p)}{(p - f'(x))^2} = a + 2bxp^2 > 0. \quad (IV')$$

* This statement seems to contradict directly the theorem on sufficient conditions given on page 118 of Osgood's article. The contradiction is, however, only apparent since Osgood makes the further assumption—see page 108—that " $F_{y'y'}(x, y, p)$ does not vanish in E " from which it follows that if $F_{y'y'}(x, y, p)$ is positive at all points of $\mathfrak{C}$ for every value of p , it is also positive at all points of a certain neighborhood of $\mathfrak{C}$, for every value of p .

The four conditions (I), (II'), (III'), (IV') are therefore satisfied; even the stronger condition

$$F_{VV}(x, f(x), p) \equiv 2a + 24bap^2 > 0. \quad (II'_a)$$

Nevertheless the line $\mathcal{C}$ does not minimize the integral I .

For if we replace the line AB by the broken line APB , the

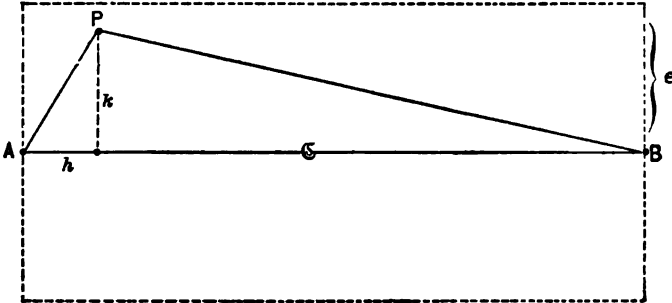


FIG. 4.

coördinates of P being $x = h > 0$ and $y = k$, the total variation of I is easily found to be

$$\Delta I = k^2 \left[-\frac{bk^2}{h^2} + \frac{a}{h} + a + 3bk^2 \right] + (h) \quad (10)$$

where $\lim_{h \rightarrow 0} L(h) = 0$.

Now let $\epsilon > 0$ be given as small as we please, then choose $|k| < \epsilon$ and let h approach zero, keeping k fixed. Then since $b > 0$ it follows that

$$\Delta I < 0$$

for all sufficiently small values of h , which proves that the line AB does not minimize the integral I .

The explanation of the fact that the four conditions in question are not sufficient in the one problem whereas the corresponding four conditions are sufficient in the other problem is as follows: In Weierstrass's problem no restriction is imposed upon the direction of the tangents of admissible curves, whereas in our problem the direction parallel to the y -axis, and no other direction, is excluded. In the former case the totality of admissible directions is closed, in the latter it is not closed.

UNIVERSITY OF CHICAGO,
June 13, 1902.

ON THE SUFFICIENT CONDITIONS IN THE CALCULUS OF VARIATIONS.

BY DR. E. E. HEDRICK.

(Read before the American Mathematical Society, December 28, 1901.)

THE sufficient conditions in the calculus of variations have recently received a great deal of attention ; * and it would seem fitting that attempts be made to simplify their discussion whenever possible, and to render the agreement more exact between the known necessary and the known sufficient conditions. Such is the purpose of this paper, which also seeks to present the sufficient conditions in compact form. The work will to a large extent follow lectures delivered at Göttingen by Professor Hilbert, 1899-1901.

1. HILBERT'S INVARIANT INTEGRAL.† WEIERSTRASS'S SUFFICIENT CONDITION.

Let us consider a simple definite line integral

$$(1) \quad I = \int_{x_0}^{x_1} f(x, y, y') dx,$$

where

$$y' = \frac{dy(x)}{dx},$$

and where f is an analytic function of the three arguments x, y, y' , in a certain region R . Let us then restrict ourselves to the consideration of curves of integration contained in a realm B , consisting of curves of the type

$$(2) \quad y = \phi(x),$$

* Du Bois-Reymond, *Math. Annalen*, 15 ; Kneser, *Lehrbuch der Variationsrechnung*, and many memoirs in *Math. Annalen* ; Osgood, *Annals of Math.*, 2d ser., vol. 2, no. 3, and *Transactions Amer. Math. Soc.*, vol. 2, pp. 166, 273 ; Whittemore, *Annals of Math.*, 2d ser., vol. 2, no. 3 ; Bolza, *Ithaca Colloquium* (summer meeting, Amer. Math. Soc., Aug., 1901), unpublished, and *Transactions Amer. Math. Soc.*, vol. 2, p. 422 ; Weierstrass, lectures at Berlin, 1879-1882, unpublished ; Hilbert, lectures at Göttingen, 1899-1901, unpublished, etc.

† Compare Osgood, *Annals of Math.*, 1. c., where Hilbert's proof is given.

where $\phi(x)$ together with its first derivative is a single valued and continuous function of x in the interval $x_0 \leq x \leq x_1$, and where $\phi(x_0)$ and $\phi(x_1)$ have certain fixed values y_0 and y_1 respectively. If the value of the integral I taken along a curve

$$(3) \quad y = y(x)$$

of the realm B be less than the value of I taken along any other curve of the realm B ,

$$(4) \quad y = F(x) \equiv y(x) + \eta(x),$$

where $F(x)$ is restricted by the same conditions as y and also by

$$(5) \quad |\eta(x)| < \delta, \quad |\eta'(x)| < \delta, \quad x_0 \leq x \leq x_1,$$

δ being a positive constant chosen at pleasure, then the curve (3) is said to render the integral I a *weak* minimum, as compared to curves of the realm B . If in place of the conditions (5) we merely require that

$$(6) \quad |\eta(x)| < \delta, \quad x_0 \leq x \leq x_1,$$

then the curve (3) is said to render I a *strong* minimum.* If in addition to the condition (5) [or (6)] we require that

$$(7) \quad \begin{aligned} \eta(x_i) &= 0; \quad x_0 + (i-2) \cdot \delta \leq x_i \leq x_0 + (i-1)\delta \leq x_1; \\ (i &= 2, 3, \dots, n, (n+1); \quad n \cdot \delta = x_1 - x_0), \end{aligned}$$

then we will say that the curve (3) renders I a *limited weak* [or *strong*] minimum.†

It can be shown ‡ that if the curve (3) is to render I a minimum (of any kind), $y(x)$ must satisfy Lagrange's equation

$$(8) \quad \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0.$$

The solutions of this equation are called *extremals*.

Let us now assume: §

* For the above, cf. Osgood, *Annals of Math.*, l. c., p. 108.

† This is what Hilbert calls "ein Minimum bei stückweiser Variation."

‡ See Whittmore, l. c., p. 130, where Hilbert's proof is reproduced.

§ Compare Osgood, *Annals of Math.*, l. c., p. 113; *Transactions Amer. Math. Soc.*, l. c., p. 168.

(a) That an extremal C ,

$$(9) \quad y = y(x),$$

can be obtained, such that $y_0 = y(x_0)$ and $y_1 = y(x_1)$, i. e., such that C passes through the two fixed points $P: (x_0, y_0)$ and $Q: (x_1, y_1)$ given in the problem.

(b) That a one parameter family of extremals

$$(10) \quad y = \phi(x, \alpha)$$

can be obtained, where ϕ is an analytic* function of x and α , $x_0 \leq x \leq x_1$, $|\alpha| < \epsilon$; and where

$$(11) \quad \phi(x, 0) = y(x).$$

(c) That

$$\frac{\partial \phi(x, 0)}{\partial \alpha} \neq 0; \quad x_0 < x \leq x_1.$$

It then follows that through any point of a suitably chosen neighborhood N about the curve C , one and only one extremal of the family (10) can be passed; and that (10) can be solved for α in terms of x and y ,

$$(12) \quad \alpha = \psi(x, y),$$

where $\psi(x, y)$ is a singly valued analytic function of x and y , in the neighborhood N . Hence also

$$y' = \frac{\partial \phi(x, \alpha)}{\partial x}$$

can be expressed as a singly valued analytic function of x and y ,

$$(13) \quad y' = p(x, y),$$

in the neighborhood N .

Hilbert now considers the integral

$$(14) \quad J = \int_{x_0}^{x_1} [f(x, y, p) + (y' - p)f_p(x, y, p)] dx$$

taken along any curve $\overline{C}$, $y = Y(x)$, joining P and Q , and

* Since f has been assumed analytic in x, y, y' , it follows that the coefficients in (8) are analytic, and hence that the solutions of (8) are analytic.

lying in the neighborhood N , where y and y' refer to the curve of integration and p denotes the above function $p(x, y)$, and where f_p means $\partial f / \partial p$. It is then shown that this integral J is independent of the path of integration,* i. e., independent of $F(x)$. It is seen at once that $J_C = I_C$, where J_C denotes the value of the integral J taken along the curve C , and so on. But $J_{\bar{C}} = J_C$ since J is independent of the path. Hence the necessary and sufficient condition for a minimum, which is that $I_{\bar{C}} > I_C$, becomes at once $I_{\bar{C}} > J_{\bar{C}}$ or $(I - J)_{\bar{C}} > 0$, or

$$(15) \quad \int_{x_0}^{x_1} E(x, y, y', p) dx \Big|_{\bar{C}} > 0,$$

where $\bar{C}$ is any curve in N other than C , and where

$$(16) \quad E(x, y, y', p) \equiv f(x, y, y') - f(x, y, p) - (y' - p)f_p(x, y, p).$$

It follows that

$$(17) \quad E(x, y, y', p) \geq 0, \quad x_0 \leq x \leq x_1$$

for all x, y near the curve C , for the function p found above, and for any y' whatever, is a sufficient condition for a strong minimum,† where the sign of equality is to hold only along the extremals (10). And, moreover, the condition

$$(18) \quad E(x, y, y', p) \geq 0, \quad x_0 < x < x_1,$$

for all x, y, p on C and for all y' whatever, is a necessary condition for a strong minimum.† For if, for simplicity, the family of extremals (10) be so taken as all to pass through $P: (x_0, y_0)$, then a curve can be found which renders the integral less than C does, in case $E < 0$ at any point of C in any direction $y' \neq p$. For we need only take as the comparison curve desired an extremal other than C , together with a curve which cuts C at this point in the given direction. The integral $\int E dx$ taken along such a curve is certainly negative.

Let us now consider a fixed point $R: (x, y)$, on the curve C , and $f(x, y, y')$ as a function of y' alone, for these fixed values of x and y . Let us then draw in a y'/f -plane the curve

* See Osgood, *Annals of Math.*, I. c., p. 122.

† Provided, of course, that conditions (a), (b), (c) hold.

$f=f(y')$. By (16) the inequality $E > 0$ becomes

$$(19) \quad \frac{f(y') - f(p)}{y' - p} \leq f_p(p) \text{ according as } y' - p \leq 0.$$

Let θ be the angle made by the axis of y' with the tangent to the y'/f -curve at the fixed point where $y' = p$. And let ψ be the angle between the y' axis and the chord joining this fixed point to a variable point where $y' = y'$. Then, as is easily seen from a figure,

$$\tan \theta = f_p(p) \quad \text{and} \quad \tan \psi = \frac{f(y') - f(p)}{y' - p}.$$

Hence the inequality (19) becomes

$$(20) \quad \tan \psi \leq \tan \theta \quad \text{according as} \quad y' \leq p.$$

But this is surely fulfilled (except, of course, for $y' = p$) provided the y'/f -curve is always concave upward, the f axis having been taken vertical, for the range of values of y' considered.

Hence the condition

$$(21) \quad \frac{\partial^2 f}{\partial y'^2} > 0,$$

for all x, y on the curve C and for all y' considered, is a sufficient condition for a minimum.* For if (21) is satisfied, the y'/f -curve for every point (x, y) , on or near $\dagger C$, will be concave upwards, for the whole range of values of y' considered, and hence, by the above, the Weierstrass sufficient condition (17) will be satisfied.†

The condition (21) is, however, by no means necessary. A necessary condition usually attributed to Legendre resembles it in form. This necessary condition requires that $\frac{\partial^2 f}{\partial y'^2} \geq 0$ for all

* Weak or strong, according as the values of y' are restricted, or not. Here, as elsewhere, the region R of page 11 must not be overstepped. In particular, the value $y' = \infty$ must always be carefully investigated.

† On account of the continuity of $\frac{\partial^2 f}{\partial y'^2}$ in x, y, y' .

‡ This also follows from $\text{Limit}_{y' \rightarrow p} \left[\frac{E(x, y, y', p)}{(y' - p)^2} \right] = \frac{\partial^2 f}{\partial y'^2} \Big|_{y' = p}$.

x, y on the curve C and for $y' = p$.* A proof of this theorem is given in the following section.

2. LIMITED VARIATION. LEGENDRE'S CONDITION.

If we now consider limited variation only, we have the following theorem: *Lagrange's, Legendre's narrower,† and Weierstrass's ‡ conditions are together sufficient for a limited strong minimum.*

We suppose first found a solution C of Lagrange's equation connecting the points P and Q . Around any point of C a region can so be bounded off that one and only one extremal exists joining the given point to any point of the region; which follows at once from the Cauchy-Kowalewski existence theorems,

since $\left. \frac{\partial^2 f}{\partial y'^2} \right|_C \neq 0$ at such a point, from the narrower condition of

Legendre. If now, we require that $\eta(x) = 0$ at least once in the interior of each of a set of regions such that the Cauchy-Kowalewski solutions starting from the one end point are unique up to the other end point, we surely have in each such region a satisfactory field for the Weierstrass-Hilbert theory. It follows that the Weierstrass-Hilbert reasoning can be applied, and hence, if the Weierstrass condition is satisfied, the curve C actually renders the integral I a limited strong minimum.

Likewise it is clear that *Lagrange's, and Legendre's narrower conditions alone are together sufficient for a limited weak minimum.* For Legendre's condition, $\left. \frac{\partial^2 f}{\partial y'^2} \right|_C > 0$ is satisfied. But this insures § the fulfillment of Weierstrass's sufficient condition, by the end of Section 1, since $\frac{\partial^2 f}{\partial y'^2}$ is a continuous func-

* For this condition we shall use the notation $\left. \frac{\partial^2 f}{\partial y'^2} \right|_C \geq 0$; and we shall call it, temporarily, Legendre's *broadier* condition; later, Legendre's *necessary* condition. The corresponding condition, $\left. \frac{\partial^2 f}{\partial y'^2} \right|_C > 0$, where the sign of equality may not hold, we shall call, temporarily, Legendre's *narrower* condition; later, *sufficient*. It is readily seen that $\left. \frac{\partial^2 f}{\partial y'^2} \right|_C > 0$ is sufficient for a weak minimum, under conditions (a), (b), (c).

† See preceding footnote.

‡ See condition (17) above.

§ For weak variation, of course.

tion of x, y, y' . On account of this theorem we will henceforth call the condition $\left[\frac{\partial^2 f}{\partial y'^2} \right]_C > 0$ *Legendre's sufficient condition*.

We will now proceed to show that Legendre's broader condition is necessary for any kind of a minimum whatever. We have seen that Legendre's sufficient condition, $\left[\frac{\partial^2 f}{\partial y'^2} \right]_C > 0$ is, together with Lagrange's condition, sufficient for a limited weak minimum. But that I have a *minimum* is equivalent to saying that $-I$ has a *maximum*. Hence $\left[\frac{\partial^2 f}{\partial y'^2} \right]_C < 0$ is, together with Lagrange's condition, a sufficient condition for a limited weak *maximum*. Given now a solution of Lagrange's equation, C , joining P and Q , if $\left[\frac{\partial^2 f}{\partial y'^2} \right]_C < 0$ at any point of C , then $\left[\frac{\partial^2 f}{\partial y'^2} \right]_C < 0$ for a whole interval, since $\frac{\partial^2 f}{\partial y'^2}$ is a continuous function of x, y, y' , and C is a continuous curve whose tangent varies continuously. But since $\left[\frac{\partial^2 f}{\partial y'^2} \right]_C < 0$ is a *sufficient condition* for a limited weak *maximum*, it follows that *any limited weak variation* in this interval gives us a curve for which I has a *less value* than along C . It follows that Legendre's broader condition, $\left[\frac{\partial^2 f}{\partial y'^2} \right]_C \geq 0$, is a *necessary condition* for a *minimum of any kind whatever*; since otherwise we have actually found a comparison curve C' such that $I_{C'} < I_C$, and this comparison curve occurs in any possible choice of realms of curves to be considered since it is a *limited weak variation* of C , the tangents to which can be made, if necessary, to turn continuously.* On account of this theorem we shall henceforth call the condition $\left[\frac{\partial^2 f}{\partial y'^2} \right]_C \geq 0$ *Legendre's necessary condition*.

It seems fitting at this point to call attention to the general usefulness of the idea of the limited variation, which has led us so easily to Legendre's necessary condition. The question

* Or even analytically, by the now well known method originally due to Schwarz.

of geodetics in the surface theory, and many questions in mechanics, notably the Hamilton-Jacobi theory, require only the use of *limited* variation, and indeed it is peculiarly suited to the needs of these and other similar subjects. Its greater simplicity in statement and treatment commend it above methods commonly employed, and its results are more easily obtained and grasped. For instance, in the surface theory, any geodetic line actually renders the integral of length on the surface a *limited* strong minimum along *any* portion of it which joins *any* two points upon it.* This statement brings into a clear light the essential characteristics of a geodetic as a "shortest" line, and its application to such special surfaces as the sphere and the anchor ring are particularly vivid.

3. UNLIMITED VARIATION. PRODUCTION OF JACOBI'S CONDITION FROM WEIERSTRASS'S CONDITION.†

Jacobi's condition was originally erroneously stated as a sufficient condition, and was shown by Weierstrass to be insufficient for strong minima. Thus far we have obtained Weierstrass's condition (Section 1) as sufficient for a strong minimum under the necessary conditions (a), (b), (c) of Section 1. We now propose to obtain Jacobi's condition as a necessary condition for any (*unlimited*) minimum from consideration and elimination of these conditions (a), (b), (c). We shall then show that Jacobi's condition (together with Lagrange's and Legendre's) is a *sufficient* condition for *weak* minima, the Weierstrass condition being unnecessary. Thus the Jacobi condition precisely covers the introduction of the *unlimited* variation, for exactly analogous results have been obtained for *limited* variation (Section 2) with its omission.

We will suppose the Lagrange condition satisfied, since it is a *first* necessary condition for any discussion. Let the supposed solution C of our problem be the curve

$$(1) \quad y = y(x)$$

joining $P: (x_0, y_0)$ and $Q: (x_1, y_1)$. The Lagrange equation

$$(2) \quad y'' \frac{\partial^2 f}{\partial y'^2} + y' \frac{\partial^2 f}{\partial y \partial y'} + \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial f}{\partial y} = 0$$

* In this particular theorem, of course, no new *fact* is involved.

† In connection with this section, see Osgood, *Transactions Amer. Math. Soc.*, l. c., p. 166.

has in general a two-parameter family of solutions

$$(3) \quad y = \phi(x, c_1, c_2)$$

where ϕ is an analytic function of x, c_1, c_2 if, as supposed, f is analytic in x, y, y' .

Let us now suppose that c_1 and c_2 are so chosen that

$$(4) \quad \phi(x, 0, 0) \equiv y(x),$$

so that for $c_1 = 0, c_2 = 0$, our curve C results. Further let $c_1 = t\gamma_1$ and $c_2 = t\gamma_2$. Then (3) becomes

$$(5) \quad y = \phi(x, t\gamma_1, t\gamma_2).$$

Since ϕ is analytic in c_1 and c_2 we have, when γ_1 and γ_2 are regarded as constant,

$$(6) \quad \begin{cases} y = \phi(x, t\gamma_1, t\gamma_2) = \Phi(x, t), \text{ say,} \\ = \Phi(x, 0) + t \frac{\partial \Phi(x, 0)}{\partial t} + \frac{t^2}{2!} \frac{\partial^2 \Phi(x, 0)}{\partial t^2} + \dots \\ = y(x) + t\phi_1(x) + t^2\phi_2(x) + \dots \end{cases}$$

which converges uniformly for sufficiently small values of t .

Now, regarding γ_1 and γ_2 as constant, and (6) as our one-parameter family of solutions required by condition (b), Section 1, let us set

$$(7) \quad p = \frac{\partial \Phi(x, t)}{\partial x} = \frac{dy(x)}{dx} + t \frac{d\phi_1(x)}{dx} + t^2 \frac{d\phi_2(x)}{dx} + \dots$$

If now we can solve (6) for t as a single valued function of y and x , we can insert the value found in (7) and have p defined as a single valued function of y and x , *i. e.*, of the position; and this will hold for sufficiently small values of t , since the series used are then uniformly convergent, *i. e.*, it will hold for all points in a suitably chosen neighborhood of C . Such a solution will then satisfy the purposes of the Weierstrass-Hilbert theory. But for this (compare condition c, Section I) it is only necessary that

$$(8) \quad \phi_1(x) \neq 0.$$

But

$$\phi_1(x) = \left. \frac{d\Phi(x, t)}{dt} \right]_{t=0},$$

whence

$$\begin{aligned} (9) \quad \phi_1(x) &= \frac{\partial \phi(x, c_1, c_2)}{\partial c_1} \frac{dc_1}{dt} + \frac{\partial \phi(x, c_1, c_2)}{\partial c_2} \frac{dc_2}{dt} \\ &= \frac{\partial \phi}{\partial c_1} \gamma_1 + \frac{\partial \phi}{\partial c_2} \gamma_2 = \eta_1 \gamma_1 + \eta_2 \gamma_2, \end{aligned}$$

$$\text{where } \eta_1 \equiv \frac{\partial \phi}{\partial c_1}, \quad \eta_2 \equiv \frac{\partial \phi}{\partial c_2}.$$

If then we can so choose γ_1 and γ_2 that $\phi(x)$ does not vanish between x_0 and x_1 the Weierstrass-Hilbert theory can be applied.

And further, the possibility of such a choice is necessary for the existence of a minimum. For, consider the family of extremals out of (3) which pass through (x_0, y_0) and suppose that γ_1 and γ_2 are so chosen that (6) represents this family.* If then $\phi_1(x)$ vanishes at any point R between x_0 and x_1 then the equations

$$(10) \quad \begin{cases} y = \Phi(x, t), \\ \frac{d\Phi(x, t)}{dt} = 0, \\ y = y(x) \equiv \Phi(x, 0) \end{cases}.$$

are satisfied at this point, since

$$\left. \frac{d\Phi(x, t)}{dt} \right]_{t=0} = \phi_1(x).$$

Hence the envelope of the above family of extremals through (x_0, y_0) cuts the curve C at any such point as R .

If now this envelope be a single point, then compare the value I_C with $I_{\bar{C}}$ where $\bar{C}$ is any other extremal of the family. Since J is independent of the path and since J reduces to I for all extremals, it follows that C does not render I a proper (unlimited) minimum of any sort whatever, between P and R or

* It can be shown from the considerations at the end of this Section, that this is the best possible choice.

between P and any point beyond R . For $I_C]_P^R = I_{\bar{C}}]_P^R$ and $\bar{C}$ is an analytic weak variation of C .

If on the other hand the envelope is a curve, then compare $I_C]_P^R$ with $I_C]_P^{R'} + I_E]_{R'}^R$, where $\bar{C}$ is any other extremal of the family, E the envelope, and R' the point where $\bar{C}$ meets E . Since the envelope is tangent at every point to an extremal of the family, it follows, as before, that $J_E = I_E$ and hence $I_C]_P^R = I_{\bar{C}}]_P^{R'} + I_E]_{R'}^R$. Hence C does not render I a proper* (unlimited) minimum between P and R nor between P and any point beyond R .

It follows that if we can so choose γ_1 and γ_2 that

$$\phi_1(x) \neq 0, \quad x_0 < x \leq x_1,$$

the Weierstrass-Hilbert theory can be applied; and further the possibility of such a choice is a necessary condition for the existence of an (unlimited) minimum of any sort.

We know that

$$(11) \quad y = \phi(x, c_1, c_2) = \Phi(x, t)$$

must satisfy Lagrange's equation

$$(12) \quad L(y) \equiv \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0$$

or

$$(13) \quad 0 = L(\Phi(x, t)) = L(y(x) + t\phi_1(x) + t^2\phi_2(x) + \dots) \\ = L(y(x)) + \frac{dL(y(x) + t\phi_1(x) + \dots)}{dt} \Big|_{t=0} t + \frac{d^2L}{dt^2} \Big|_{t=0} \frac{t^2}{2!} + \dots$$

for all sufficiently small values of t . Hence the coefficient of t must vanish separately,

$$(14) \quad \frac{d}{dt} \{L(y(x) + t\phi_1(x) + t^2\phi_2(x) + \dots)\} \Big|_{t=0} = 0.$$

But $L(\xi)$ involves $\xi, \frac{d\xi}{dx}, \frac{d^2\xi}{dx^2}$. Hence we have

* It can indeed be shown that C does not even render I an improper minimum, for the envelope cannot be an extremal. It is to be noticed that all the above work holds even when the extremals cut the envelope.

$$(15) \quad \frac{\partial L(\xi)}{\partial \frac{d^2 \xi}{dx^2}} \frac{d^2 \xi}{dx^2} \frac{d}{dt} + \frac{\partial L(\xi)}{\partial \frac{d \xi}{dx}} \frac{d \xi}{dx} \frac{d}{dt} + \frac{\partial L(\xi)}{\partial \xi} \frac{d \xi}{dt} \Big]_{t=0} = 0,$$

where

$$\xi \equiv y(x) + t\phi_1(x) + t^2\phi_2(x) + \dots;$$

or

$$(16) \quad \frac{\partial L(y)}{\partial y''} \phi_1''(x) + \frac{\partial L(y)}{\partial y'} \phi_1'(x) + \frac{\partial L(y)}{\partial y} \phi_1(x) = 0,$$

an equation which must be satisfied by $y = y(x)$. Using the definition of $L(y)$, we find by a simple calculation that $\phi_1(x)$, and hence both η_1 and η_2 , must satisfy the following linear differential equation of the second order:

$$(17) \quad f_{y'y'} \cdot \eta'' + \frac{df_{y'y'}}{dx} \eta' + \left(\frac{df_{y'y'}}{dx} - f_{yy} \right) \eta = 0,$$

where

$$\eta'' \equiv \frac{d^2 \eta}{dx^2}, \quad f_{y'y'} \equiv \frac{\partial^2 f}{\partial y'^2}, \quad \text{etc.,}$$

as usual. But (17) is precisely Jacobi's equation; and Jacobi's whole condition consisted in finding whether an integral of this equation

$$\phi_1(x) = \gamma_1 \eta_1 + \gamma_2 \eta_2$$

existed, which did not vanish between x_0 and x_1 . Hence the above are precisely Jacobi's conditions, except that the equality sign in our condition must be omitted in the usual discussion, and the corresponding case made a subject of further investigation.

The rest of the results announced at the beginning of the article now follow readily, and will be left to the reader.

4. SUMMARY OF CONDITIONS.

Collecting our results, we may say that the following conditions hold:

	Limited Variations.		Unlimited Variations.	
	Weak.	Strong.	Weak.	Strong.
<i>Necessary.</i>	Lagrange's, Legendre's necessary.	Lagrange's, Legendre's necessary, Weierstrass's necessary.	Lagrange's, Legendre's necessary, Jacobi's.	Lagrange's, Legendre's necessary, Jacobi's, Weierstrass's necessary.
<i>Sufficient.</i>	Lagrange's, Legendre's sufficient.	Lagrange's, Legendre's sufficient, Weierstrass's sufficient.	Lagrange's, Legendre's sufficient, Jacobi's.	Lagrange's, Legendre's sufficient, Jacobi's, Weierstrass's sufficient.

It is seen on glancing at the table that from the simple conditions (Lagrange's and Legendre's) for limited weak variation we proceed to any other case by adding Weierstrass's conditions in the case of a strong minimum, and Jacobi's in case of an unlimited minimum, only. It is to be hoped that advances may be made in bringing the necessary and sufficient conditions more closely together or into entire coincidence. The above table represents *substantially* the present known conditions, in the belief of the writer.

In special problems the irksomeness of these conditions can sometimes be circumvented. For instance, *given a problem in*

which $\frac{\partial^2 f}{\partial y'^2} > 0$ for all values of x, y, y' , then the necessary and

sufficient condition for a limited strong minimum is the possibility of finding a solution of Lagrange's equation joining the two given end points. Such is the case in the geodetic problem and also in the integral which leads to Hamilton's principle; and in each of these cases, fortunately, a limited strong minimum is all that is desired. Similar simplification

occurs in every case when $\frac{\partial^2 f}{\partial y'^2} > 0$ for all x, y, y' . For then

Legendre's and Weierstrass's conditions are always satisfied, and may be abstracted from the above table. For this reason

Hilbert has called a problem in which $\frac{\partial^2 f}{\partial y'^2} > 0$ for all x, y, y'

contained in a singly connected region R , in which the given end points lie, a "regular" problem of the calculus of variations.

5. CONCLUSION. HILBERT'S EXISTENCE THEOREM.

A misunderstanding has sometimes arisen in regard to Hilbert's existence proof. Its purpose is not to establish further conditions nor to solve more general problems, but is to reassure us as to the possibility of satisfying the conditions already found, in the more favorable cases. We have seen that a *necessary* condition for any minimum was the possibility of finding a solution of Lagrange's equation joining the two end points P and Q ; and that this is a *sufficient* condition for a limited strong minimum, in a *regular* problem. That such a solution exists, even in the most favorable cases, is by no means certain from the ordinary theory of differential equations. It was the original purpose of Hilbert's proof (merely) to show that, in this most favorable case of a regular problem, a curve must exist which renders our integral an unlimited strong minimum, compared with all continuous comparison curves; and that the minimizing curve is composed of a finite number of pieces of extremals. Hilbert's existence theorem may therefore properly be called a theorem in differential equations.

At present the results cannot be said to demonstrate more than that a curve exists which renders the integral an *improper* minimum. This result *cannot* be extended in general, and it remains to show that Lagrange's equation is necessary, not only for proper, but also for improper minima.

In conclusion, the author desires to enter protest against the extreme complication recently introduced in some quarters into the essentially simple subject of the calculus of variations. In the case of the only modern text-book on the theory,* this condition is so exaggerated as to essentially mar the usefulness of the book, in that many who would otherwise interest themselves in the subject, are repelled by the style and treatment. It is to be sincerely regretted, in the opinion of the writer, that this tendency has been followed in some of the recent memoirs.

YALE UNIVERSITY,
June, 1902.

* Kneser, Variationsrechnung.

SOME RECENT BOOKS ON MECHANICS.*

Vorlesungen über technische Mechanik. Von AUGUST FÖPPL. 4 Bde., 8vo. Leipzig, B. G. Teubner. Bd. 1: *Einführung in die Mechanik*, 2te Aufl., 1900, xiv + 422 pp.; Bd. 2: *Graphische Statik*, 1900, x + 452 pp.; Bd. 3: *Festigkeitslehre*, 2te Aufl., 1900, xviii + 512 pp.; Bd. 4: *Dynamik*, 2te Aufl., 1901, xv + 506 pp.

Einführung in das Studium der theoretischen Physik, insbesondere in das der analytischen Mechanik, mit einer Einleitung in die Theorie der physikalischen Erkenntniss. Von P. VOLKMANN. Leipzig, B. G. Teubner, 1900. xvi + 370 pp.

Quelques Réflexions sur la Mécanique, suivies d'une première Leçon de Dynamique. Par EMILE PICARD. Paris, Gauthier-Villars et Fils, 1902. 56 pp.

FROM the standpoint of the teacher who has to face the problem of introducing students to the elements of mechanics both theoretical and practical, probably the most interesting and suggestive work which has appeared during the last few years, is that of Dr. August Föppl, professor at the school of technology (Technische Hochschule) at Munich. The books are especially noteworthy because Professor Föppl teaches those who are to be engineers—practical men—and yet at the same time insists on teaching in such a manner that his pupils shall acquire not the applications alone but a firm grasp of the general theoretical principles which underlie those applications. This method of instruction unfortunately may not appeal to some in this country who care only for what our engineers can do and not at all for what they may know. But certainly it is much better to teach the thought and theory in addition to the routine and practice, or in conjunction with it. In fact, as Professor Föppl points out, the engineer can not be thoroughly master of his work and capable of handling whatever new difficulties may arise unless he knows the why as well as the how.

What then is the sort of lectures that a thorough scientist finds feasible when addressing a class of engineering students at one of the leading German scientific schools? This question

* Continued from the June number of the BULLETIN.

may be answered by looking into Professor Föppl's four volumes—for happily the author has followed his lectures closely. That the work is found practical and acceptable in Germany is shown clearly by the demand for a second edition within about two years of the time of first publication. In this review we shall lay especial stress on the first and fourth volumes, which are less technical than the second and third and which consequently are more interesting to us here.

The first volume, *Introduction to Mechanics*, discusses the elementary portions of the kinetics and statics of particles and rigid bodies with or without friction, of the theory of elasticity, and of hydromechanics. Throughout, the object of the author is to give his pupils a general knowledge of mechanical theory such that later the more advanced or special branches will be seen to fit on in their proper places. The language is natural and simple. The only mathematical prerequisite is a fair knowledge of differential and integral calculus. Every attempt is made to relieve the reader from difficulties of presentation in order that he may concentrate his undivided attention upon the new difficulties of the subject itself.

There is one possible exception to this statement. The author throughout all four volumes uses vector analysis wherever it is appropriate. The notations are those which he has previously employed in his *Einführung in die Maxwell'sche Theorie der Electricität*. The vectors are printed in heavy faced German type. If we may be permitted to change to English Clarendons, let $\mathbf{a}$, $\mathbf{b}$ represent two vectors and $\mathbf{e}$ a vector of unit length normal to $\mathbf{a}$ and $\mathbf{b}$ upon that side of the $\mathbf{ab}$ -plane upon which rotation from $\mathbf{a}$ to $\mathbf{b}$ appears counterclockwise. Then the scalar and vector products of $\mathbf{a}$ and $\mathbf{b}$ are defined as

$$\mathbf{a}\mathbf{b} = -\mathbf{S}\mathbf{a}\mathbf{b} = \mathbf{a} \cdot \mathbf{b} = [\mathbf{a}|\mathbf{b}] = ab \cos(\mathbf{a}, \mathbf{b}),$$

$$\mathbf{V}\mathbf{a}\mathbf{b} = \mathbf{V}\mathbf{a}\mathbf{b} = \mathbf{a} \times \mathbf{b} = [\mathbf{a}\mathbf{b}] = ab \cos(\mathbf{a}, \mathbf{b}) \mathbf{e}.$$

(The notations are those of Föppl, Sir W. R. Hamilton, Gibbs, and Grassman in order.) Although the author had found in his own experience that vector methods were better suited to pedagogical purposes than cartesian analysis, he had grave fears at first lest the public at large, especially in Germany, should disagree with him and dislike his method. Yet he went boldly ahead and published it. The result was a little surpris-

ing to him. Those who used the volumes were pleased, and even the reviewers could not make objections without adding the qualifying statement that: "The presentation of the chief general theorems of mechanics in terms of vectors possesses a simplicity and intelligibility which could not be obtained by the old method of cartesian coördinates." The great popularity of Professor Föppl's work seems to bear out this judgment and to indicate further that the use of vectors certainly does not increase the difficulty of learning mechanics.

What style shall be adopted in presenting elementary mechanics is one of the most difficult questions an author or a teacher must meet. The students' previous training in algebra, trigonometry, analytic geometry, and calculus as it is generally taught has been necessarily quite formal. These mighty algorithms of formal mathematics must be learned so that they may be applied with readiness and precision. But with mechanics comes the application of these algorithms and formal do-by-rote methods, though often possible, yield no results of permanent value. How to elicit and cultivate thought is now of primary importance. There are two opposite methods. First. Give the student a syllabus of the subject in which the general theory, the bare facts, are presented with as little discussion as possible. The student must then supply his own discussion, his own details. He must think the subject over until he sees its meaning. If he has patience, ability, and a little guidance the result is good. Otherwise he gets little or nothing. Second. Enter into a discussion of the subject, touch on this side and on that, show what is known and what is not, point out where definitions are needed, where experiment must be appealed to, and thus by merely dwelling on the different aspects of the subject allow the true significance of it all to become apparent to the student. This method is often very successful in introducing new ideas such as mass, time, and force. It is practically taking them as innate ideas after talking long enough to insure that this innateness has been recognized. But it also has its disadvantages. For unless it is well handled the student is apt to be bewildered by the lack of definiteness, and to lose faith in his ability to work accurately. It may be remembered that we spoke favorably of Professor Gray's *Treatise on Physics* in that he struck a mean between these two methods.

Professor Föppl, writing for a totally different class of readers,

makes an uncommonly successful use of the second—the discursive method. The discussion is half historical, half scientific, such as is found much more elaborately worked out in Professor E. von Mach's *Science of Mechanics*, and is sure to appeal to one who commences mechanics with his common sense alert. The introduction to Volume I, a short essay on the origin and aim of mechanics, is excellent diet for future engineers and physicists. The author points out what is meant by experience—true, useful experience—and shows how the fact that geometry was given a definitive, logical form as long ago as the time of Euclid may be interpreted not as proving that geometry is not ultimately founded on experience, but rather as indicating how long the experimental stage antedated Euclid. To give mechanics a definitive form is now possible, as the work of Hertz and others shows. Some say that this straightforward logical method of presentation is the *simplest*. That depends on what is meant by *simple*. From the standpoint of the learner the historical crude method appears easier than the logical. Here Professor Föppl touches either rightly or wrongly on a great pedagogical principle. Perhaps it is interesting and not unsuggestive to note that while teachers and authors in our universities are trying to give calculus a logical rigor and mechanics a definitive set of axioms, there is in the meantime a great commotion among teachers of elementary geometry as to whether this logical definitive method of Euclid is at all suitable for beginners. Some scientists of note, Professor John Perry and Mr. Oliver Heaviside, for example, have taken part in the discussion and placed themselves emphatically upon the negative. As far as mechanics is concerned Professor Föppl seems to agree with them.

After the introduction, twelve pages of the text are devoted to general discussion of the ideas involved in particle, inertia, and force. The author does not define; he prefers to discuss. His treatment of a particle is particularly noteworthy. On page 17 we read: "The bodies or parts of bodies which we shall regard as particles may be reasonably small, but must not be very (*i. e.*, unlimitedly) small." The earth is a particle for many problems; a molecule for few. Setting a lower limit for the size of a particle in dynamics is usually forgotten; yet, sooner or later, it is of the greatest importance if students of atomic theories are not to be confused. A particle must generally be regarded either as composed of a large number of molecules or as a small part of an atom.

We might point out further that the common definition of density at a point as

$$D = \frac{dm}{dv}$$

is quite meaningless even in the case of a so-called homogeneous body unless we make an assumption which is never physically correct and which need never be made, namely that mass is a continuous, not to say differentiable function of position. Might it not be better to tell the student, especially one who is to be a physicist, that density and a large number of physical quantities are averages — averages which are taken over a region so large in comparison to the phenomenon averaged that the result is sufficiently stable to be considered as a definite number for practical purposes, but over a region so small with respect to the total region in which the phenomenon exists that the methods of the differential calculus may be employed in expressing and handling the average? Whether it is better to make physical hypotheses which are ultimately untenable in order that in the meantime mathematics may be applied with absolute rigor or to state that the methods of pure mathematics, though strictly not applicable, yield results which within negligible errors agree with the phenomena observed or observable? Professor Föppl seems to believe that discussion made in the right spirit is, all things considered, more satisfactory to practical men than precise definitions made on impossible physical hypotheses.

The law of inertia is explained and illustrated by Foucault's pendulum experiment. The illustration is subtle but striking. The fact that the pendulum will beat on in its own absolute plane regardless of the earth is particularly impressive and quite intelligible to students. Force is then discussed qualitatively; but it is only after the treatment of the motion of freely falling bodies that quantitative definitions of force and mass are given. Experiment is called upon to establish the fact that the acceleration g , due to gravity, is constant at a particular point in the surface of the earth. The force which the earth exerts on a body is called the weight of that body. Weights may be compared and a unit adopted. Mass may be introduced by means of the equation

$$W = mg = m \frac{d^2x}{dt^2}.$$

Finally by making the assumption that any force is essentially like the force due to gravity in that it may be expressed numerically in terms of weight units the author arrives simultaneously at a measure of force in general and at the fundamental dynamical equation for motion in one dimension—that is, the expression

$$F = ma = m \frac{d^2x}{dt^2}.$$

Thus far only rectilinear motion has been treated. The author now proceeds to motion in two or three dimensions, to the parallelogram law, and the principle of the superposition of different motions. The discussion accorded to centrifugal force is perhaps unnecessarily long and is concerned mainly with showing how that force should be regarded as the equal and opposite reaction to the centripetal force, not by any means as a fictive force. Work and energy are introduced and the fundamental theorem that the work done is equal to the change in kinetic energy is proved. The author introduces the principle of virtual velocities and proves it from the parallelogram law. He states, however, that the inverse method of procedure is equally admissible. The principle of virtual velocities may be taken as the fundamental physical axiom instead of the parallelogram law which may then be proved from it. Finally a long treatment of moments and the solution of a number of problems complete the discussion of the dynamics of a particle—the first division of the first volume.

The second division deals with the mechanics of rigid bodies. After elementary kinematical considerations, statics and the principle of virtual velocities are taken up. This part of the work has been regarded with great disfavor by some reviewers, notably by Dr. K. Heun in the first number of the *Zeitschrift für Mathematik und Physik* for the current year. The statement which gives trouble is found on page 143: "Let us give to a body an arbitrary virtual displacement. This displacement can even be finite. It is, however, customary to apply the principle of virtual velocities in the case of a rigid body only to infinitesimal changes of position and we shall consequently derive it for such. The transference to a finite displacement offers no difficulty inasmuch as each finite displacement can be reduced to a sum of infinitesimal changes of position." If by "can even be finite," the author means can in *all* cases be finite,

as Dr. Heun seems to imagine, the statement is certainly in gross error; for it makes the principle of virtual velocities applicable only to cases of neutral equilibrium. But we see no reason for not interpreting the statement as meaning can in *some* cases be finite. In fact we are wont to use the principle of virtual velocities, as far as practical applications are concerned, mainly for finding the mechanical advantage and efficiency of machines. When a machine such as an inclined plane, a screw, or a pulley system is working at a constant rate the forces acting upon it must be in neutral equilibrium. We are inclined to believe that in a larger number of the problems in which virtual velocities are useful to the engineering student the forces must likewise be in neutral equilibrium. We might suggest that the author express himself a little more clearly on this point for the sake of those who seem to have misunderstood him. There are other points upon which the author might easily be defended from his reviewers if necessity made a defense advisable.

Before leaving the principle of virtual velocities we wish to ask whether the introduction of that subject into elementary mechanics is to be commended. Almost all English and some continental books practically ignore it. When it is not ignored the treatment given it is usually indefinite and unsatisfactory (Professor Föppl's book forms no exception) owing to the fact that the necessary and sufficient conditions as to rigidity, constraints, internal and frictional actions which must be satisfied in order that the principle be applicable are not clearly stated.

The third division of the volume deals first with the statical, second with the dynamical properties of centers of gravity, and finally with the kinetic energy of a rigid body. From this point of departure the fourth division leads naturally through the undissipated transformation of energy to the fifth division which is a long and practical treatment of frictional or dissipative forces. Sliding, rolling, axial, and string or belt frictions are each discussed carefully with applications.

In the sixth division the student is made acquainted with some of the fundamental ideas of elasticity and strength of materials which he will use in the third volume; in the seventh, with the direct and oblique impact of a perfectly elastic or perfectly inelastic sphere on another sphere or on a plane; in the eighth, with those qualitative and simpler quantitative facts of hydromechanics which are deducible without a great knowledge

of either mechanics or mathematics. It might have been better if Professor Föppl had included in the seventh division a discussion of imperfectly elastic impact. There is plenty of room as the volume contains only about four hundred pages.

The second volume, *Graphical Statics*, is about four hundred and fifty pages long and is divided into seven parts: composition and resolution of forces acting at a point or in a plane, the funicular polygon, forces in space, trusses in a plane, trusses in space, elastic deformation of trusses and statically undetermined trusses, theory of arches.

The inclination toward theoretical in addition to practical considerations is indicated in the treatment given to such subjects as the reciprocal relations between force-plans, the differential equations of a funicular curve under a load anywise distributed, the null system, the long analytic discussion of the theorem that a truss which contains only the necessary number of members and is stable must be statically determined and conversely is stable if it is statically determined for all possible loads. We miss, however, both in this volume and in the next, the interesting theorem that under any given conditions the most economical truss is one which is statically determined. We also happily miss the elaborate plates which generally accompany a work on graphical statics and which add so enormously to the price. The figures Professor Föppl gives are simple, comparatively inexpensive, and incorporated in the text. The object of the author is to give the reader a usable figure and allow him to draw one more accurate if he desires. Students of theoretical mechanics would do well to learn as much graphics as is contained in this work.

The third volume, *Resistance or Strength of Materials*, is also for the most part as instructive to those interested in theoretical mechanics as to engineers: for the author has not filled his book with masses of detail and tables of constants, but has dealt with the deduction of those theoretical results which must be obtained before the tables are usable. The eleven divisions are entitled; general investigations on stress, elastic deformation and specification of materials, bending of a straight bar, energy of deformation, bars with a curved line of center, bars on a yielding foundation, strength of plane plates supported along their contour, strength of vessels under external and internal pressures, torsional strength, buckling strength, and the elements of the mathematical theory of elasticity. The tendency of this

volume, like the others, to develop the theory of the subject would probably be distasteful, if not unintelligible, to our engineers in this country.

Owing to the nature of the subject exact mathematical developments, even when possible, would lead to expressions too complicated for use. Professor Föppl therefore introduces approximations very freely, and in many cases discusses the maximum error due to these approximations. There is one case where we fear his assumptions may not be quite justifiable. In solving the problem of a torsion of a rectangular beam he assumes that the strain is expressible by the first three terms of a power series. He omits, however, to test his results by means of the stress equations. The complete solution is due to St. Venant and is expanded into a series of transcendental terms. The last division gives a satisfactory and not very difficult introduction to the general mathematical theory of elastic isotropic bodies. This is practically the only elaborate addition to the lectures actually delivered by the author to his students. Even this would be given in the regular course, he states, were it not for sheer lack of time!

With the fourth volume, Dynamics, the author returns to more purely theoretical discussions. In fact, the first and fourth volumes, with the last division of the third, form a complete treatise on the elements of mechanics of a particle, a rigid body or a system of rigid bodies, of fluids, whether perfect or not, and of elastic bodies. The titles of the five divisions of this last volume are: dynamics of a particle, dynamics of a rigid body and a swarm of particles, relative motion, dynamics of systems of bodies (Lagrange's equations, etc.), hydrodynamics. The discussion is mainly a completion and extension of that in the first volume.

In the first division, central motion, potential, and harmonic vibrations are treated. The fact that the author does not overlook forced vibrations is noteworthy. Electrical developments of the last fifteen years have caused forced vibrations to grow constantly more and more important. Yet in few, even of the recent text-books, is any mention made of the subject. The treatment of the conservation of moment of momentum in the second division is enlivened by such interesting examples as the fall of a cat, the shrinkage of the earth, the ability to accelerate one's motion while swinging, and the effect on the motion of a ship due to the rotation of the machinery. The motion of a

top, impulses applied to a rigid body, and the theory of oscillations are some of the other subjects treated. The third division offers two proofs of Coriolis's theorem, and discusses some applications of relative motion.

In the fourth division that too great carelessness in stating restrictions, which was mentioned earlier in connection with the presentation accorded to the principle of virtual velocities in the first volume, obscures and mars the development of Lagrange's equation and Hamilton's principle. The method in which generalized forces are introduced is hardly the best, hardly convincing. A discussion of mechanical similarity and its applications closes the division. The treatment of hydrodynamics, like that of the mathematical theory of elasticity, is a valuable introduction to the subject. Perhaps they both are more nearly abstracts than introductions. For this includes, in addition to Euler's equations, the motion of a sphere through a perfect fluid, Helmholtz's theorem, the equations of a viscous fluid, and that contains not merely the fundamental equations, but also to a considerable extent the theories of Boussinesq, Hertz and St. Venant.

In conclusion let us state that the value of each volume is materially enhanced by the summary which concludes it and contains a list of the principal formulæ. The third volume has already been published by Gauthier-Villars in a French translation. We could wish that the entire work might be translated into English with the hope that it might stimulate our teachers and students of engineering to be more thorough and complete in their theoretical considerations. But it is doubtful whether the venture would prove much of a financial success. Engineers would think the work insufficiently practical. Physicists would think it too practical. In differing with the former we should call attention to that suggestive little book "Made in Germany," which was published not long since in England for the purpose of setting forth the reasons why Germany was coming so rapidly to the front in all lines of practical endeavor. One of the chief reasons given was the German's thoroughness and determination to know his field from beginning to end. In differing from the physicists we should point out that in those two volumes—the second and third—which they would probably scorn most, the name of Maxwell appears frequently. The presence of that name might aid in convincing students of theoretical and practical physics that they could

doubtless find quite as much profit in becoming acquainted with the contents of these volumes as in examining the more abstruse and unpractical parts of their science.

Professor Volkmann's Introduction to Theoretical Physics is a protest against the tendency toward the more or less artificial presentations given to mechanics by Kirchhoff, Hertz, and even Boltzmann. It is an attempt to show how by harking back to Newton and the English method of treating mechanics a greater clearness, a no less deep insight, and with sufficient care a no less scientifically accurate presentation may be obtained. The whole work is such a curious medley of history, philosophy of scientific method, foundations of mechanics, theoretical mechanics, and applications that we are at a loss to know for what sort of readers it can be intended. For those who already do know the subject the applications are not necessary; for those who do not, the philosophy is confusing. The author's explanation to his readers that "You have already busied yourselves to a certain extent with physics and mechanics and thereby commence these lectures with certain ideas and expectations which may be just as much of a help and value as of a hindrance and harm to you," seems on the whole to be of little use in trying to decide who his readers may be. We fear Professor Volkmann has fallen into an altogether too common German habit of writing to please himself.

The first division of the book is an introduction to the theory of physical knowledge and methods of physical thought. There are forty-four pages of octavo text with numerous footnotes referring to Newton, Kant, Lotze, Thomson and Tait, Helmholtz, and especially Volkmann. Then follows a bibliography of twenty-six titles of which ten are to previous articles by the author. To criticise the author's theory which is itself presented in compact form would take too much space. The theory is interesting to read; but how valuable it may be to either a theoretical or practical physicist is not clear.

The second division, about seventy pages, systematically lays the foundations for mechanics according to Galileo and Newton and works out the application to the case of a particle. The author postulates a conception of time, mass, and a universal ether which is to act as a fundamental system of reference. Just what these conceptions are we are unable to determine; but the necessity of postulating them apparently becomes evi-

dent after reading the author's discussion, if indeed it were not evident before. Next, the ideas of inertia, vector velocity, vector accelerations, and physical dimensions are introduced and applied to the motion of a projectile in vacuo.

As a foundation for kinetics, Newton's laws are stated and discussed critically. The analytic statement of the laws and the development of the dynamics of a free or constrained particle complete this division of the book. We are unable to ascertain precisely what a particle (*Massenpunkt*) is. The term does not appear in the index and a definition seems lacking in the text. Fortunately, however, we are told in due season that the postulates of space, time, and mass are: There is a *space* of entirely determinate geometric characteristics in which reality transpires—the euclidean space. There is a *time* which flows on uniformly and is independent of all phenomena. There is a quantity—the *mass*—which preserves its identity unchanged through all changes of phenomena. The word *phenomenon* has been used in its etymological meaning as a translation for *Erscheinung*.

The third division develops the mechanics of a system of particles whether the distribution be discontinuous or continuous. Especial attention is given to the general theorems of motion of the center of gravity, rate of change of moment of momentum, and energy. The student ought to be able to get a firm grasp on these fundamental principles and be ready to understand their application to those various vibrating physical instruments which are discussed theoretically in the fourth division.

In the fifth division, after twenty pages of preliminary remarks on fluids and how to treat them, the author settles down to hydrostatics first in its usual aspect of phenomena on the interior of a fluid and then in that part of the subject which takes into account the effects of surfaces of discontinuity and is more often known as the theory of capillary action. The sixth division contains such questions introductory to geophysics as the attraction of spheres, the relation of gravitation and weight, methods of determining the gravitation constant G and the density of the earth.

The discussion of the general principles of mechanics forms the seventh and last division of the work. The author had here an opportunity to collate and compare the various general principles—those of D'Alembert, Lagrange, Gauss, Maupertuis, and Hamilton. This opportunity he accepted. The result can

hardly be considered any material improvement over other treatments whether in definiteness or completeness. The volume ends with an index which is an absolute necessity for a book of such varied propensities.

If we have given an intelligible account of the contents of Professor Volkmann's work we shall feel satisfied. To review it connectedly is next to impossible. We had hoped that we might find in it a careful, precise, intelligible and connected treatment of theoretical physics or analytic mechanics. The precision and connection, at least, are for the most part lacking. This very lack, however, furnishes to those who are acquainted with mechanics, a stimulus to thought and thus the book may become more interesting and suggestive than one less indefinite — which is about as much as we can say for it.

Professor Picard's little volume consists of two parts—an essay on Mechanics and the Science of Energy and a First Lesson in Dynamics. The former is merely an extract from the general report on science which the author wrote in connection with the Exposition in 1900. The latter was delivered in 1894 and has hitherto been accessible only in periodicals. That they are now printed together ought to be a cause for congratulation among students and teachers of mechanics.

The first noticeable feature of the work is the perfect, the transcendent clearness of the style. It is unnecessary to inquire for whom the book was written. Any one can read it with ease, pleasure and profit.

The author shows how the idea held at the end of the eighteenth century that mechanics had attained a definite form has proved illusory, and how modern scientists, especially Hertz, have been able to reduce mechanics to a more logical set of axioms and a more entirely deductive form. Yet clearly behind a great admiration for Hertz and his work there stands out the evident feeling on the part of the author that something more practical, more anthropomorphic is quite as convenient and valuable.

There is one point which seems a trifle misleading. Upon page 14 we find the statement that dynamics of a particle will probably not become obsolete for the reason that atomic hypotheses play and will perhaps always play a preponderating rôle in science. The reason is not convincing. When an atom is treated as a definite object, it is not necessarily and not usually

regarded as a particle. It is either a rigid or a fluid body. A particle, as was pointed out above on page 28 of this review, may be a large aggregate of atoms or a small part of an atom, but not often the atom itself.

In his discussion of what is meant by the possibility of a mechanical explanation of all natural phenomena, Professor Picard points out that some mean the possibility of reducing the phenomena to one type of differential equations or another; some, the existence of a universally applicable principle like that of least action; and some, the construction of a dynamical model, as an example of which Kelvin's model of the ether may be mentioned.

The chapter on the science of energy states the different points of view of two classes of scientists, of which one believes energy to be nothing else than an abstract conception without real existence and of which the other believes in the real existence of different kinds of energy commutable into each other. Particular attention is given to the relation of known experimental results to the various laws of energy. Mention of the law of dissipation affords an opportunity to discuss whether or not that law is compatible with a mechanical explanation of the universe.

Professor Picard's first lesson in dynamics contains numerous departures from other introductory presentations of the subject. His particle is something small enough to be regarded without negligible error as a geometric point. This seems unnecessarily indefinite and yet unnecessarily definite. The important thing is that the motion does not differ appreciably from translation. Whether the particle may be regarded as a point has little to do with it. Inertia is presented in the Newtonian manner. Force is the name for any action which causes motion.

With the introduction of the measure of force there appears the first novelty. The author commences with a *field of force*. In particular, a *constant* field of force is such that particles left to themselves at different points of the field describe congruent paths in the same manner. The acceleration in a constant field is constant and may be taken as the measure of the field. Constant fields, or any fields at a definite point, may be compounded according to the parallelogram law. If force be defined statically by means of any dynamometer, experiment will show that the static and dynamic measures of force are proportional. In the same field, for example in the field of gravity

near the earth's surface, the forces acting on two masses are proportional to the masses. The unit of force or mass may therefore be chosen arbitrarily. The lesson closes with the determination of the fundamental differential equation of motion for a variable force and the discussion of work done by a force.

The mental acumen of a student who can understand and appreciate the whole of this, his first lesson, must be considerable. No further comment is necessary on this presentation of the fundamental ideas, which seems sufficiently accurate and modern, yet quite practical, and withal not far from Newton's classic treatment.

We have now passed in review about a dozen recent volumes on mechanics. Many different standpoints are represented, and often represented very well. But what is more noteworthy and valuable is the steady general improvement over older works, not only in the matter presented, but in the method of presentation. There is yet much to be done in producing treatises on mechanics. Perfection has not been reached, nor even nearly enough approached. There is one field in particular in which a thorough and careful work is greatly to be desired. That is the field of general dynamical theory. Sooner or later—let us hope it will be sooner—there must appear a work which shall take such general principles as D'Alembert's, Lagrange's and Hamilton's, and state just what are the restrictions under which each may be applied in practice. For instance, in Lagrange's *Mécanique Analytique* we find at least four distinct ways of writing the general equations of motion. Each has its peculiar advantages and restrictions. Who will set these forth? There is also plenty of room for a brief, almost syllabic, treatise on dynamical theory. Few students are able to appreciate how extremely limited purely theoretical considerations are in dynamics. Six or eight hours is quite sufficient to carry a lecturer from the beginning through the derivation of Lagrange's equations and Hamilton's principle, if all applications be omitted. And it is only thus that a thoroughly comprehensive grasp of dynamics can be obtained.

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
June 10, 1902.

ON A NEW EDITION OF STOLZ'S ALLGEMEINE
ARITHMETIK, WITH AN ACCOUNT OF
PEANO'S DEFINITION
OF NUMBER.

Theoretische Arithmetik. Von O. STOLZ (Innsbruck) und J. A. GMEINER (Wien). I. Abtheilung: *Allgemeines. Die Lehre von den rationalen Zahlen.* Leipzig, B. G. Teubner, 1901. iv + 98 pp.

THIS publication of about 100 pages is the first instalment of a new and revised edition of Stolz's *Allgemeine Arithmetik* (1885-86),—a work which has long since proved indispensable to all who desire a systematic and rigorous development of the fundamental elements of modern arithmetic.

The revision thus far completed (probably about one seventh of the entire work) covers the first four chapters:

I. On quantities (Größen) and operations (Verknüpfungen) in general.

II. On the natural numbers and the four fundamental operations.

III. On the general properties of any direct operation (Thesis, $a \circ b$) and its inverse (Lysis, $a \sim b$), as deduced from certain fundamental formal laws; in particular, the analytic theory of (absolute) rational numbers.

IV. On the synthetic theory of (absolute) rational numbers, with a treatment of systematic fractions.

The remaining chapters in the first part will contain (if the order in the old book is preserved) the theory of negative and irrational numbers, with an account of euclidean "ratios," followed by an elaborate treatment of the theory of limits as applied to functions of a real variable and to infinite series of real terms. The second part will then contain the theory of operations on complex numbers, including chapters on infinite series, infinite products and continued fractions. The complete work will belong to the Teubner series of mathematical textbooks.

As one turns the pages of the new edition one is struck first of all by the great improvement in the general appearance of the book. The title itself, "theoretical arithmetic," is much

more lucid than the old title of "general arithmetic," which to the uninitiated might mean common algebra; the notes, historic and other, which used to be difficult of reference, are now placed conveniently as footnotes on the proper pages; and best of all the paragraph numbers in the several sections are now placed at the top of each page. All these changes will be appreciated by those who have had to refer often to the older book. The addition of some thirty problems or exercises is also very welcome.

In the body of the text there are numerous changes which greatly enhance the clearness of the presentation. The introduction of the rational numbers (III, 7) and the opening paragraph of IV should be especially mentioned. We notice that in III no attempt is made to prove the mutual independence of the postulates $A) - E)$, and as a matter of fact if $E, 2)$ is made to read: "when $b > b'$ then $a \circ b > a \circ b'$ " the postulate D_1 can be at once deduced as a theorem.

The only radical change from the plan of the first edition, however, is found in the second chapter, where the treatment of the natural numbers is now based on Peano's definition. The details of Peano's theory are here for the first time made accessible to readers unfamiliar with his mathematical language. Since, however, no systematic treatment of Peano's method is obtainable in English, it may be not without interest to devote the remainder of our space to a connected account of its principles.

PEANO'S THEORY OF THE NATURAL NUMBERS.*

The fundamental concepts which form the starting-point of Peano's theory are three: (1) a class or assemblage, denoted by N ; (2) a special object, denoted by E ; (3) what we may call a "rule of succession," (denoted by $\circ$), according to which every object, a , of the given class N determines uniquely another object, $a \circ$, called the successor of a .

The nature of the assemblage N , of the special object E , and of the rule of succession $\circ$ is left wholly undetermined, except for the following five restrictions, or postulates:

* G. Peano: *Arithmetices principia nova methodo exposita*, Turin, 1889; "Sul concetto di numero," *Rivista di Matematica*, vol. 1 (1891), pp. 87-102, 256-267; *Formulaire de Mathématiques*, vol. 2 (1898), no. 2, pp. 1-59; vol. 3 (1901), pp. 39-44. Where I have written N and E , Peano uses N_1 and 1 or later N_0 and 0; and for $a \circ$, Peano uses $a +$ or later $a + 1$. I have also changed the wording of postulate 5. Stolz and Gmeiner use "Zahl," 1 and $a + 1$, for N , E and $a \circ$, and add another postulate, to the effect that every number shall be called equal to itself.

1. *The special object E shall belong to the class N .*
2. *If an object a belongs to N then its successor $a \circ$ shall also belong to N .*
3. *If an object a belongs to N then its successor $a \circ$ shall be different from E .*
4. *If $a \circ = b \circ$ then $a = b$.*
5. *If a is any given element of N , then a can be found in the sequence $E, E \circ, (E \circ) \circ, \dots$*

The meaning of these postulates will be clearer, if we consider at once the following systems, in which N , E and $\circ$ are so chosen that all the postulates, or all but one, are satisfied.

- (a) N = class of positive integers; $E = 1$; $a \circ = a + 1$.
- (b) N = class of positive integers with 0; $E = 0$; $a \circ = a + 1$.
- (c) N = class of positive integral powers of 2;

$$E = 2^1; a \circ = 2a.$$

Each of these systems (a), (b), (c) satisfies all the five postulates.

- (S_1) N = class of positive integers; $E = 0$; $a \circ = a + 1$.
- (S_2) N = class of positive integers from 1 to 9;

$$E = 1; a \circ = a + 1.$$

- (S_3) N = class of positive integers from 1 to 9;

$$E = 1; a \circ = a + 1, \text{ except that } 9 \circ = 1.$$

- (S_4) N = class consisting of two elements 0 and 1;

$$E = 0; 0 \circ = 1, 1 \circ = 1.$$

- (S_5) N = class of positive integers; $E = 1$; $a \circ = a + 2$.

Here the system S_k ($k = 1, 2, 3, 4, 5$) satisfies all the other postulates, but not the k th.

The possibility of constructing such systems as (a), (b), (c) proves that *the postulates 1-5 are consistent*; and the consideration of the systems S_1 - S_5 shows us that *the postulates 1-5 are independent of one another*. This proof of independence, by the way, which forms a most interesting feature of Peano's work, is not reproduced by Stolz and Gmeiner.

Further, it can be easily shown that any two systems, S and S' , which satisfy all the postulates 1-5 can be brought into one-to-one correspondence with each other in such a way that

E will correspond with E' , and whenever a in S corresponds with a' in S' then $a \circ$ will correspond with $a' \circ$. This fact may be stated in the theorem: *All systems $(N, E, \circ)$ which satisfy postulates 1-5 are equivalent to one another* (cf. the systems $a-c$ above). On the basis of this theorem Peano then states his definition:

A "system of natural numbers" shall mean any system $(N, E, \circ)$ which satisfies the postulates 1-5.

It is clear that such a definition is a very different thing from a psychological analysis of the process of counting. It is rather the construction of a symbolic "calculus of operations," which contains indeed the arithmetic calculus as a special case when a special interpretation, viz., (a) , is given to its symbols, but in which the validity of the deductions is quite independent of any such interpretation.

The first step in the development of the theory is the definition of the symbol $a + b$ as follows: Let a and b be any two natural numbers, that is, any two elements of a system $(N, E, \circ)$ which satisfies Peano's five postulates; then $a + b$ denotes the number determined by the following recursion formulæ:

$$\begin{aligned} a + E &= a \circ; & a + (E \circ) &= (a + E) \circ; & \dots; \\ a + (k \circ) &= (a + k) \circ. \end{aligned} \tag{6}$$

Thus, in order to compute the value of $a + b$ we must compute in succession the values of

$$a + E, \quad a + (E \circ), \quad a + [(E \circ) \circ], \quad \dots, \tag{6'}$$

until we strike the desired element $a + b$.

The statement of Stolz and Gmeiner, defining $a + b$ as that number which is obtained by starting from a and proceeding b steps forward in the series of numbers, would seem to imply that the *ability to count* is after all involved in this process. Such is, however, not the case. In order to find the number $a + b$ by Peano's definition we require simply the ability to determine from any given element a the element $a \circ$; we can thus compute one after the other the elements of the sequence $(6')$, and we know by postulate 5 that this sequence will contain the element $a + b$. How far we shall have to go to find this element we do not know; we do not count the steps already taken at any stage, nor do we ask how many more there are to

take; it is sufficient to know that if we persevere, one step at a time, the desired number will surely be reached.*

The element $a + b$ thus determined is called the "sum" of the given elements a and b ; when the system bears the interpretation (α) then $a + b$ will clearly be the sum of a and b in the ordinary sense.

Similarly, the "product," ab , of two elements a and b is defined by the recursion formulæ

$$aE = a; \quad a(E\circ) = (aE) + a; \quad \dots; \quad a(k\circ) = (ak) + a. \quad (7)$$

From the definition of $+$ follows next the theorem

$$a + (b + c) = (a + b) + c$$

the proof of which we give in full as a typical illustration of the use of postulate 5 (method of mathematical induction). Thus: the theorem is obviously true when $c = E$, by (6). And if it is true when $c\circ$ is any particular element k then it will be true also when $c = k\circ$.† Therefore, being true for $c = E$, it is true for $c = E\circ$, and hence for $c = (E\circ)\circ$, etc. But by 5 any given element can be reached in this way; hence the theorem is true for all values of c .

In a similar way the theorems

$$a + b = b + a, \quad (ab)c = a(bc), \quad ab = ba,$$

and

$$(a + b)c = ac + bc$$

are established.

The definition of the symbols $>$ and $<$ depends on the following theorem, which is proved also by the use of (6): If $a \neq b$ then there is either a number λ such that $a = b + \lambda$ or else a number μ such that $b = a + \mu$ (and not both). In the first case a is called "greater than" b ($a > b$), in the second case "less" ($a < b$). (8)

When the system bears the interpretation (α) the use of the symbols $>$ and $<$ as thus defined will clearly agree with the ordinary use of them.

* Of course this implies unlimited time at our disposal—an assumption not explicitly stated by Peano or by Stolz and Gmeiner.

† For on the given assumption we have, by 6,

$$(a + b) + (k\circ) = [(a + b) + k]\circ = [a + (b + k)]\circ = a + [(b + k)\circ] = a + [b + (k\circ)].$$

From the definition of $>$ and $<$ we have at once that $a + b > a$, and also the theorems: If $a > a'$ then $a + b > a' + b$ and $ab > a'b$. For example, from $a > a'$ follows $a = a' + x$, whence $ab = (a' + x)b = a'b + xb$, or $ab > a'b$. This proof of Peano's is simpler than the induction proof used by Stolz and Gmeiner.

After these fundamental definitions and theorems have been established the further development of the theory—including the introduction of the rational, irrational, negative and complex numbers—proceeds along lines already familiar and need not be enlarged upon here.

In a sense, the five postulates of Peano may be said to contain, implicitly, all the results of algebra and analysis, including the theory of functions of a complex variable. It should be clearly understood, however, that the business of the mathematical theory is solely to draw logical deductions from these postulates, not to discuss the question whether the results possess any real significance in the objective world. On this deeper question, Peano's work merely reduces the problem to its lowest terms; if we can show that his fundamental postulates are capable of real interpretation in the objective world, then all their consequences will likewise be capable of such interpretation. Whether, however, a real interpretation of the postulates—for example, the system (α) on p. 42)—is psychologically possible or not, is a problem of epistemology with which the mathematical theory has nothing whatever to do.

A precisely similar remark applies to any deductive theory such as that contained in Hilbert's *Grundlagen der Geometrie* (1899). At the basis of Hilbert's theory stand certain symbols such as "point," "line," "between," etc., whose meaning is left wholly undetermined except for the imposition of certain postulates. The development of the theory consists in deducing the theorems which follow from the fundamental postulates by the laws of formal logic, quite independently of any special interpretation of the undefined symbols. To be sure, when the undefined symbols are interpreted as meaning point, line between, etc., in the ordinary sense then the whole theory becomes identical with our ordinary geometry of space. But the question whether such an interpretation is possible or not is a question which the deductive theory leaves wholly untouched. Hence Hilbert's theory cannot be said to give us any new knowledge of the real nature of space, any more than

Peano's theory gives us any new knowledge of the real nature of the natural numbers. Both these theories, however, enable us to state the philosophical problem with a definiteness which has not heretofore been possible.

Further comment on Stolz and Gmeiner's book is impossible in the space at our disposal. The remaining instalments of this great work will be awaited with keen interest.

E. V. HUNTINGTON.

HARVARD UNIVERSITY,
June 12, 1902.

LAZARUS FUCHS.*

FUCHS is dead. This announcement must have caused deep sorrow in the heart of many American mathematicians. For many of us have been his pupils, and to some of us his example has been the greatest inspiration of our lives. The writer of this little sketch is one of these. He remembers how he looked forward to the time when he would be fitted to attend Fuchs's lectures. He remembers the small and crowded lecture-room in the University of Berlin, poorly ventilated, stuffy and hot in the summer days, but so full of meaning and inspiration to the earnest and thoughtful student. Fuchs was not a brilliant lecturer. He spoke in a quiet, undemonstrative manner, but what he said was full of substance. To the student there was the inspiration of seeing a mathematical mind of the highest order full at work. For Fuchs worked when he lectured. He was rarely well prepared, but produced on the spot what he wished to say. Occasionally he would get lost in a complicated computation. Then he would look around at the audience over his glasses with a most winning and child-like smile. He was always certain of the essential points of his argument, but numerical examples gave him a great deal of trouble. He was fully conscious of this failing, and I remember well one occasion when, after a lengthy discussion, he laid considerable emphasis upon the fact that "*in this case*, two times two is four."

The mathematical papers of Fuchs are very numerous, but excepting a few of his earliest attempts, they are all connected directly or indirectly with the theory of linear differential equations. This was the province which, to quote the words of Auwers when he introduced Fuchs to the Berlin Academy of

* Immanuel Lazarus Fuchs, born in Mosochin, near Posen, May 5, 1833, died at Berlin, April 26, 1902.

Sciences, he had added to the mathematical kingdom. Of course the conquest of this new territory had been prepared by others. The theory of functions, which was an essential prerequisite had been built up by Cauchy and Riemann. The work of Briot and Bouquet on differential equations of the first order was an illustration of the application of this new theory to the treatment of differential equations. But to Fuchs belongs the credit of first applying the theory of functions to the linear differential equations of any order, with rational coefficients. His paper in the sixty-sixth volume of *Crelle's Journal* is a classic, and to this day I know of no clearer exposition of the fundamental principles involved. It is true that Riemann was acquainted with these principles, as his posthumous paper on this subject proves. It is true also that Fuchs took his immediate inspiration from Riemann's famous paper on the hypergeometric series. But all of this does not lessen the credit due to Fuchs. To generalize is one of the functions of the mathematician. This Fuchs did. Riemann also did this, but his paper was never published until the theory had long been developed by Fuchs. It is interesting to notice in this connection the difference between the points of view adopted by Riemann and Fuchs. Fuchs starts out with the linear differential equation of the n th order whose coefficients are rational functions of x . By a rigorous examination of the convergence of the series, which formally satisfies the differential equation he finds that these equations have a very important and characteristic property. The singular points of the solutions are fixed, i. e., they are independent of the constants of integration and can be found without first integrating the differential equation. They are in fact included among the poles of its coefficients. He then shows that a fundamental system of solutions undergoes a linear substitution with constant coefficients when the variable x describes a circuit enclosing such a singular point, and from this behavior of the solutions derives expressions for them, valid in a circular region surrounding that point and reaching as far as the next singular point. He thus establishes the existence of systems of n functions, uniform, finite and continuous, except in the vicinity of certain points, and undergoing linear substitutions with constant coefficients when the variable x describes circuits around these points.

Riemann's point of view was exactly opposite to this. With him it was a matter of principle not to define a function by an

expression, but by a characteristic property. He, therefore, starts out with the assumption of a system of n functions uniform, finite and continuous, except in the vicinity of certain arbitrarily assigned points, and undergoing an arbitrarily assigned linear substitution when the variable x describes a closed path around such a point. He then shows that such a system of functions will satisfy a linear differential equation of the n th order. But the theorem that the branch points and the substitutions may be arbitrarily assigned, ought to be proved at the outset if this point of view is to be adopted. Much has since been done on this question, which is really a fundamental one in the theory of linear differential equations.

This is not the place to go into details. Suffice it to say that the theory of linear differential equations was placed by Fuchs upon a solid and adequate foundation. He and his followers have reared upon this a noble structure. He himself characterized a class of such equations, called after him Fuchsian, whose solutions are everywhere regular. He studied the question of algebraic integrability which has so many points of contact with other questions of importance. He studied by his own methods the periods of an elliptic integral as function of the modulus, for Legendre had shown that these verify a linear differential equation of the second order. By considering the modulus inversely as a function of the quotient of the periods, a uniform automorphic function, now known as the modular function, was obtained. The theory of modular functions, and more generally of automorphic functions owes much to Fuchs, as well as to Klein and to Poincaré, who as an indication of this fact has named a large class of such functions Fuchsian. Fuchs has introduced other transcendental functions into analysis, connected with a linear differential equation in much the same way as the abelian functions are connected with an algebraic equation. Little has yet been done with these beyond the proof of their existence.

We have already mentioned that one of the first results obtained by Fuchs in his classical memoir was the fact that the branch points of a linear differential equation are fixed. Fuchs himself was the first to ask the question; are there other equations of this kind? In a beautiful paper in the *Sitzungsberichte der Berliner Akademie*, he started the investigation of this important question, confining himself, however, to differential equations of the first order. Poincaré completed the investiga-

tion in a remarkable manner, the result being that no essentially new functions could be defined by differential equations of the first order with fixed singular points. Painlevé has since then found that transcendental functions essentially new can be defined by such equations of order higher than the first.

We will close this brief sketch by translating a sentence, which is as characteristic of the modern theory of differential equations, as the famous definition of Kirchhoff is of modern mechanics. Fuchs says in his famous paper in *Crelle's Journal*: "In the present condition of science it is not so much the problem of the theory of differential equations to reduce a given differential equation to quadratures, as to deduce from the equation itself the behavior of its integrals at all points of the plane, i. e., for all values of the complex variable."

This is the present point of view in the theory of differential equations. The first chapter of this theory, that of linear differential equations, has been far advanced, although not completed, by Fuchs and his pupils. Something has been done on later chapters, but not much. The theory of non-linear differential equations is one of the central problems of modern mathematics, but it has not yet found its Fuchs.

E. J. WILCZYNSKI.

UNIVERSITY OF CALIFORNIA,
BERKELEY, July 8, 1902.

NOTES.

THE fifty-first annual meeting of the American association for the advancement of science, was held at Pittsburgh, June 28-July 3. The total number of members in attendance was 431, and 320 papers and addresses were presented, of which 24 are classified under mathematics and astronomy (Section A). Among the officers elected for the coming year are: President of the Association, Dr. IRA REMSEN; Vice-president of Section A, Professor G. B. HALSTED; Secretary of Section A, Professor C. S. HOWE. A report of the meeting of Section A will appear in an early number of the BULLETIN. It was decided to hold the next meeting of the Association at Washington, D. C., in convocation week, December 29, 1902-January 3, 1903.

IN its annual list of doctorates conferred by American universities during the academic year closing with June, 1902, *Science*

records the following with theses in mathematics: A. B. COBLE, Johns Hopkins University, "A study of the ternary quartic in its relation to the conics"; OTTO DUNKEL, Harvard University, "Regular singular points of a system of homogeneous linear differential equations of the first order"; PETER FIELD, Cornell University, "The forms of unicursal quintic curves"; CARL GUNDERSEN, Columbia University, "On the measure or content of assemblages of points"; C. J. KEYSER, Columbia University, "The plane geometry of the point in space of four dimensions"; T. M. PUTNAM, University of Chicago, "Concerning the linear fractional group on three variables with coefficients in the Galois field of order p "; H. L. RIETZ, Cornell University, "On primitive groups of odd order"; W. E. TAYLOR, Syracuse University, "On the product of an alternant by a symmetric function."

OXFORD UNIVERSITY. — The following courses in mathematics are announced for Michaelmas term, 1902: — By Professor E. B. ELLIOTT: Theory of numbers, two hours; Theory of functions, one hour. — By Professor H. H. TURNER: Elementary mathematical astronomy, two hours. — By Professor W. ESSON: Analytic geometry of plane curves, two hours; Synthetic geometry of plane curves, one hour. — By Professor A. E. H. LOVE: Spherical harmonics and other methods of analysis that are appropriate in applications to physics, three hours. — By Mr. C. E. HASELFOOT: Algebra, two hours. — By Mr. C. LEUDES-DORF: Projective geometry, three hours. — By Mr. A. E. JOLLIFFE: Analytical geometry, two hours. — By Mr. J. W. RUSSELL: Differential calculus, two hours. — By Mr. R. F. McNEILE: Curve tracing, one hour. — By Mr. P. J. KIRKBY: Introduction to higher algebra, one hour. — By Mr. A. L. PEDDER: Problems in pure mathematics, one hour. — By Mr. C. H. SAMPSON: Solid geometry, two hours. — By Mr. J. E. CAMPBELL: Differential equations, two hours. — By Mr. C. H. THOMPSON: Integral calculus, two hours. — By Mr. E. H. HAYES: Analytical statics, three hours. — By Mr. A. L. DIXON: Hydrostatics, one hour. — By Mr. H. T. GERRANS: Advanced rigid dynamics, two hours.

Considerable changes will be made in the First Public Examination for those who seek honors in mathematics by a scheme which comes into effect in 1904. A new subject, viz:

The elements of analytical geometry of three dimensions, is to be introduced ; and restrictions on the freedom of choice of methods are to be removed. Under the existing scheme the elements of the mechanics of solids and fluids is treated without the aid of the infinitesimal calculus ; in future the use of the calculus in its applications to mechanics is to be encouraged. At present there is a separation of geometry into two papers, devoted respectively to analytical and synthetic methods ; in future candidates are to be encouraged to choose the method, analytical or synthetic, which appears the more appropriate in answering any question. The examination is taken in the first year or the second year of the academic course, and successful candidates have the option of proceeding to a final honors school in mathematics, or in physics, or in any other subject ; in these respects the new scheme makes no changes.

THE several German universities below offer during the winter semester, 1902-1903, courses in mathematics as follows :

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ : Differential calculus, four hours, with exercises, two hours ; Maxima and minima in elementary geometrical treatment, two hours ; Applications of the theory of elliptic functions, four hours ; Colloquium, two hours ; Seminar, three hours. — By Professor G. FROBENIUS : Theory of determinants, four hours ; Algebra, four hours ; Seminar, three hours. — By Professor R. LEHMANN-FILHÉS : Analytical geometry, four hours ; Theory of probabilities, one hour. — By Professor J. KNOBLAUCH : Theory of space curves and curved surfaces, four hours ; Analytical mechanics, four hours ; Proseminar, one hour. — By Professor K. HENSEL : Integral calculus, four hours, with exercises, two hours ; Theory of elliptic functions, four hours. — Professor G. HETTNER : Theory of definite integrals, two hours. — By Dr. E. LANDAU : Theory of numbers, four hours ; Theory of linear differential equations, four hours ; Mengenlehre, one hour.

UNIVERSITY OF BONN. — By Professor L. HEFFTER : Analytical mechanics, four hours ; Plane analytical geometry, four hours, with exercises, one hour. — By Professor H. KORTUM : Elliptic functions, four hours ; Seminar, two hours. — By Professor R. LIPSCHITZ : Integral calculus II, four hours ; Seminar, two hours. — By Professor J. SOMMER : Algebra, two hours ; Geometrical applications of the theory of functions,

two hours. — By Dr. WENTSCHER : Philosophy of mathematics, one hour.

UNIVERSITY OF Breslau. — By Professor J. ROSANES : Differential calculus and elements of integral calculus, four hours ; Theory of invariants, two hours ; Seminar, one hour. — By Professor R. STURM : History of mathematics, one hour ; Seminar, two hours ; Theory of numbers, three hours ; Geometrical loci of higher degree, three hours. — By Professor F. LONDON : Introduction to the theory of differential equations, three hours ; Mathematics of insurance, two hours.

UNIVERSITY OF Erlangen. — By Professor P. GORDAN : Analytical geometry, four hours ; Invariants, four hours ; Seminar, three hours. — By Professor M. NÖTHER : Infinitesimal calculus, four hours ; Theory of functions, four hours. — Seminar.

UNIVERSITY OF Freiburg. — By Professor J. LÜROTH : Analytical mechanics, four hours ; Least squares, two hours ; Seminar. — By Professor L. STICKELBERGER : Plane analytical geometry and differential calculus, five hours, with exercises, two hours. — Theory of analytic functions, three hours. — By Professor A. LOEWY : Theory of curves and surfaces, four hours ; Mathematics of insurance, two hours. — By Dr. E. REBMANN : History of arithmetic, two hours.

UNIVERSITY OF Göttingen. — By Professor F. KLEIN : Encyclopædia of mathematics, four hours ; Seminar (with Professor Bohlmann), two hours. — By Professor D. HILBERT : Differential and integral calculus II, four hours ; Mechanics of continua, four hours ; Seminar in the theory of functions, two hours. — By Professor K. SCHWARZSCHILD : Celestial mechanics, three hours, with exercises, two hours. — By Professor H. MINKOWSKI : Algebra, three hours ; Analysis situs, two hours ; Seminar in the theory of functions, two hours. — By Professor M. BRENDel : Introduction to theoretical astronomy, three hours ; Life and work of Gauss, one hour. — By Professor F. SCHILLING : Analytical theory of curved lines and surfaces, four hours, with exercises, two hours. — By Professor G. BOHLMANN : Theory of probabilities, two hours ; Theory of risks, two hours ; Seminar, two hours ; Seminar in the mathematics of insurance, two hours. — By Dr. E. ZERMELO : Theory of functions, four hours ; Exercises in integral calculus and potential theory, two hours. — By Dr. BLUMENTHAL : Abelian func-

tions, two hours; Introduction to higher mathematics for students of science, with exercises, three hours.

UNIVERSITY OF HEIDELBERG. — By Professor L. KOENIGSBERGER: Higher algebra, theory of algebraic equations, four hours; Elements of the theory of differential equations, two hours; Calculus of variations, one hour; Theory of numbers, one hour; Seminar, two hours. — By Professor H. VALENTINER: Theory of orbits, three hours. — By Professor M. CANTOR: Differential and integral calculus, four hours, with exercises, one hour; Political arithmetic, two hours. — By Professor F. EISENLOHR: Theoretical optics, four hours; Differential and integral calculus, five hours; Potential, two hours. — By Professor K. KOEHLER: Synthetic geometry of space, three hours. — By Professor G. LANDSBERG: Descriptive geometry, with exercises, four hours; Theory of functions, three hours. — By Dr. K. BOEHM: Theory of partial differential equations, one hour; Vector analysis, one hour; D'Alembert's memoir on dynamics, one hour.

UNIVERSITY OF JENA. — By Professor J. THOMAE: Application of infinitesimal calculus to geometry, four hours; Definite integrals and Fourier series, four hours; Seminar, two hours. — By Professor A. GUTZMER: Analytical geometry of space, four hours; Analytical mechanics, four hours, with seminar, one hour. — By Professor J. FREGE: Differential and integral calculus II, with exercises, five hours.

UNIVERSITY OF KIEL. — By Professor L. POCHHAMMER: Elements of the theory of numbers, three hours; Introduction to the theory of functions, three hours; Seminar one hour. — By Professor P. HARZER: Celestial mechanics, three hours. — By Professor P. STÄCKEL: Integral calculus, three hours; Differential geometry of curved surfaces, three hours; Intrinsic geometry of curves, one hour; Seminar, one hour.

UNIVERSITY OF LEIPSIK. — Professor SCHEIBNER offers no lectures during the winter semester. — By Professor C. NEUMANN: Differential and integral calculus, four hours; Seminar, two hours. — By Professor H. BRUNS: Theory of probabilities, four hours; Celestial mechanics, two hours. — By Professor A. MAYER: Calculus of variations, four hours. — By Professor O. HÖLDER: Elliptic functions, four hours; Partial differential equations, two hours; Seminar, one hour. — By Professor F.

ENGEL: Determinants and algebraic equations, four hours; Theory of transformation groups (continuation), two hours; Seminar, two hours. — By Dr. F. HAUSDORFF: Analytical mechanics, three hours, with exercises, one hour. — By Dr. H. LIEBMANN: Analytical geometry of space, two hours; Theory of definite integrals, two hours; Descriptive geometry, two hours, with exercises, one hour.

UNIVERSITY OF STRASSBURG. — By Professor T. REYE: Geometry of position, three hours; Analytical mechanics, two hours; Seminar, two hours. — By Professor H. WEBER: Partial differential equations of mathematical physics, four hours; Selected chapters of algebra, two hours; Seminar, two hours. By Professor F. ROTH: Algebraic analysis and determinants, three hours; Analytical geometry of space, two hours; Ordinary differential equations, two hours. — By Professor A. KRAZER: Infinitesimal calculus, four hours; Analytical geometry of the plane, three hours; Seminar, two hours.

THE London mathematical society awarded its De Morgan medal for 1902 to Professor A. G. GREENHILL.

PROFESSORS ORMOND STONE, of the University of Virginia, E. H. MOORE, of the University of Chicago, and FRANK MORLEY, of Johns Hopkins University, have been appointed by the executive committee of the Carnegie Institution, as advisors in relation to original research in mathematics.

PROFESSOR W. E. STORY, while retaining his chair in Clark University, has been appointed head of the mathematical department in the newly established Clark College, Worcester, Mass. Mr. F. H. HODGE, at present fellow in mathematics in Clark University, becomes instructor in mathematics in the college.

AT the University of Nebraska, Dr. R. E. MORITZ, who recently returned from a year's study abroad, has been appointed to an adjunct professorship of mathematics.

DR. CHARLES W. M. BLACK, instructor in mathematics in the University of Oregon, died August 11, at La Grande, Ore.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- BELTRAMI (E.).** Opere matematiche, pubblicate per cura della Facoltà di scienze della r. Università di Roma. Vol. I. Milano, Hoepli, 1902. 4to. 22 + 438 pp. With portrait. Fr. 25.00
- BERTRAND (J.).** Eloges académiques. Nouvelle série: Poincaré, Cosson, Chasles, Cordier, Paris, Cauchy, Tisserand, Viète, Galilée, D. Papin, Clairaut, Euler, d'Alembert et Lagrange, Abel, Galois, Faraday, Pasteur. Avec un éloge historique de Joseph Bertrand, par G. Darboux. Paris, Hachette, 1902. 16mo. 51 + 411 pp. Fr. 3.50
- BIANCHI (L.).** Lezioni di geometria differenziale. 2a edizione, riveduta e considerevolmente aumentata, in due volumi. Vol. I. Pisa, Spoerri, 1902. 8vo. 524 pp. Fr. 18.00
- BOLZA (O.).** Concerning the geodesic curvature and the isoperimetric problem on a given surface and proof of the sufficiency of Jacobi's condition for a permanent sign of the second variation in the so-called isoperimetric problems. Printed from Vol. 9 of "The decennial publications." Chicago, University of Chicago Press, 1902. 4to. 2 + 7 pp. (University of Chicago decennial publications.) \$0.25
- BONNESEN (T.).** Analytiske studier over ikke-Euklidisk geometri. Kjöbenhavn, Ursius, 1902. 8vo. 103 pp. M. 3.60
- BOURDON (B.).** La perception visuelle de l'espace. Paris, Schleicher, 1902. 8vo. 446 pp. (Bibliothèque de pédagogie et de psychologie, Vol. 4.)
- BRIOSCHI (F.).** Opere matematiche, pubblicate per cura del Comitato per le onoranze a F. Brioschi. (In 4 volumi.) Vol. II. Milano, Hoepli, 1902. Folio. 456 pp. Fr. 25.00
- BURKHARDT (H.).** See JAHRESBERICHT.
- CAPELLI (A.).** Istituzioni di analisi algebrica. 3a edizione, con aggiunte delle lezioni di algebra complementare ad uso degli aspiranti alla licenza universitaria in scienze fisiche e matematiche. Napoli, Pellerano, 1902. 8vo. 19 + 714 pp.
- COMBESBIAC (G.).** Calcul des triquaternions. Nouvelle analyse géométrique. (Thèse.) Paris, Gauthier-Villars, 1902. 4to. 122 pp. Fr. 5.00
- COULON (J.).** Sur l'intégration des équations aux dérivées partielles du second ordre par la méthode des caractéristiques. (Thèse.) Paris, Hermann, 1902. 4to. 127 pp.
- CZUBER (E.).** Probabilités et moyennes géométriques. Ouvrage traduit de l'allemand par H. Schuermans. Paris, Hermann, 1902. 8vo. 250 pp. Fr. 8.50
- DARBOUT (G.).** See BERTRAND (J.).
- EMORY (F. L.).** See LAPLACE (P. S.).

- EPSTEIN (S.). Untersuchungen über lineare Differentialgleichungen vierter Ordnung und die zugehörigen Gruppen. Zürich, 1901. 8vo. 56 pp. M. 2.50
- FREYCINET (C.). The philosophy of mathematics. [Russian translation.] 2d edition. St. Petersburg, 1902. 8vo. 175 pp.
- GEISSLER (K.). Die Grundsätze und das Wesen des Unendlichen in der Mathematik und Philosophie. Leipzig, Teubner, 1902. 8vo. 8 + 417 pp. M. 14.00
- GERHARDT (E.). Die theoretische und praktische Bedeutung der arithmetischen Mittelsumme. (Diss.) Tübingen, 1901. 8vo. 20 pp.
- GLASER (S.). Untersuchung der Flächen dritten Grades, welche bei der Abbildung nach dem Prinzip der reziproken Radienvektoren wieder in sich selbst zurückkehren. I. (Progr.) Berlin, Gaertner, 1902. 4to. 29 pp. M. 1.00
- GMEINER (J. A.). See STOLZ (O.).
- GOLLER (A.). Ueber die Steinersche Fläche. (Progr.) München, Kellner, 1902. 8vo. 3 + 69 pp. M. 1.00
- GOUSAT (E.). Cours d'analyse mathématique (cours de la Faculté des sciences de Paris). Vol. I: Dérivées et différentielles, intégrales définies; développements en série; applications géométriques. Paris, Gauthier-Villars, 1902. 8vo. 6 + 620 pp. Fr. 20.00
- GULDBERG (A.). Ueber Integralinvarianten und Integralparameter bei Berührungs-Transformationen. Christiania, 1902. 8vo. 10 pp. (*Videnskabselskabets Skrifter*, I. Mathem.-naturvidensk. Klasse, 1902, No. 5.)
- . Ueber die Maxima und Minima der Integrale, die eine kontinuierliche Gruppe gestatten. Christiania, 1902. 8vo. 10 pp. (*Videnskabselskabets Skrifter*, I. Mathem.-naturvidensk. Klasse, 1902, No. 7.)
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The following dates have been fixed for the meetings of the Society :

	A. M.		A. M.
Sat.,	Oct. 25, 1902, 11:00	Sat.,	Feb. 23, 1903, 11:00
M.-Tu.,	Dec. 29-30, " "	"	April 25, " "

The Meeting of December 29-30 will be the Annual Meeting of the Society for the Election of Officers and Delivery of the Presidential Address.

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OF MATHEMATICAL SCIENCE

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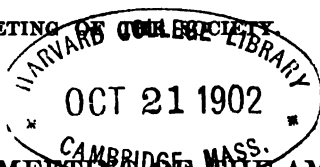
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THE NINTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE ninth summer meeting of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Ill., on Tuesday and Wednesday, September 2-3, 1902. Morning and afternoon sessions were held on each day in Room 7 of the University Hall. The President of the Society, Professor Eliakim Hastings Moore, presided at the first session, and was relieved at the later sessions by Professor T. S. Fiske and Professor H. S. White.

The attendance numbered about fifty-five, including the following thirty-seven members of the society :

Professor Oskar Bolza, Mr. A. R. Crathorne, Professor L. E. Dickson, Professor L. W. Dowling, Professor John Eiesland, Dr. William Findlay, Professor T. S. Fiske, Dr. W. B. Fite, Dr. J. W. Glover, Professor G. W. Greenwood, Professor A. S. Hathaway, Professor T. F. Holgate, Dr. Edward Kasner, Dr. H. G. Keppel, Professor Kurt Laves, Dr. D. N. Lehmer, Professor Heinrich Maschke, Professor J. A. Miller, Professor E. H. Moore, Professor Frank Morley, Dr. F. R. Moulton, Professor H. B. Newson, Professor Alexander Pell, Professor D. A. Rothrock, Miss I. M. Schottenfels, Professor G. T. Sellew, Professor J. B. Shaw, Professor E. B. Skinner, Professor Ormond Stone, Professor E. J. Townsend, Professor H. W. Tyler, Professor C. A. Waldo, Professor H. S. White, Professor W. H. Williams, Professor J. W. A. Young, Mr. J. W. Young, Professor Alexander Ziwet.

The Council announced the election of the following persons to membership in the Society: Professor T. J. I'a. Bromwich, Queen's College, Galway, Ireland; Mr. J. S. Brown, New York City; Professor G. C. Edwards, University of California, Berkeley, Cal. Eight applications for membership were received.

The Council decided that the name of the new section, authorized at the last meeting, should be the San Francisco Section; and the programme committee of the Section was approved. In accordance with a motion adopted by the Society, the Council appointed a special committee consisting of Professors Tyler (chairman), Fiske, Osgood, Young and Ziwet to

report on standard definitions of requirements in mathematical subjects for admission to colleges and scientific schools. The committee will coöperate with the corresponding committees of the Society for the promotion of engineering education, the National educational association and other interested bodies.

Besides the scientific sessions the members gathered together on various social occasions. In particular may be mentioned a dinner at the old academy building of the University, followed by a visit to the Observatory on invitation from the director, Professor Hough. Before adjourning, the Society by a rising vote expressed to Northwestern University and to the local members of the Society its appreciation of their hospitable efforts in providing for the meeting and the entertainment of the members.

The large attendance and the character of the programme is worthy of comment in view of the fact that this summer meeting is the first one held west of Buffalo.

The following papers were read at this meeting :

(1) DR. F. R. MOULTON : "A method of constructing general expressions for the elements of the planetary orbits which are valid for a finite time."

(2) PROFESSOR A. S. HATHAWAY : "The quaternion treatment of the problem of three bodies."

(3) DR. J. V. COLLINS : "A general notation for vector analysis."

(4) PROFESSOR L. E. DICKSON : "Definitions of a linear associative algebra by independent postulates."

(5) PROFESSOR L. E. DICKSON : "Definitions of a field by independent postulates."

(6) DR. E. V. HUNTINGTON : "Definitions of a field by sets of independent postulates."

(7) DR. OTTO DUNKEL : "Regular singular points of a system of homogeneous linear differential equations of the first order."

(8) PROFESSOR OSKAR BOLZA : "Some instructive examples in the calculus of variations."

(9) PROFESSOR J. B. SHAW : "On linear associative algebras."

(10) DR. W. B. FITE : "Concerning the commutator subgroups of groups whose orders are powers of primes."

(11) MR. J. W. YOUNG : "On a certain group of linear substitutions and the functions belonging to it" (preliminary report).

(12) PROFESSOR JACOB WESTLUND: "On the class number of the cyclotomic field $k(e^{2\pi i/p^n})$."

(13) PROFESSOR L. E. DICKSON: "Announcement of new simple groups" (preliminary communication).

(14) PROFESSOR ALFRED LOEWY: "Ueber die Reducibilität der Gruppen linearer homogener Substitutionen."

(15) PROFESSOR H. S. WHITE: "A special twisted cubic with rectilinear directrix."

(16) PROFESSOR F. MORLEY: "Orthocentric properties of the plane n -line."

(17) DR. VIRGIL SNYDER: "Forms of sextic scrolls of genus greater than 1."

(18) PROFESSOR PETER FIELD: "Quintic curves for which $p = 1$."

(19) DR. C. J. KEYSER: "Concerning the line and plane geometries of point four-space, and allied theories."

(20) PROFESSOR ARNOLD EMCH: "A linkage for $u = e^{i\theta}z$."

(21) DR. EDWARD KASNER: "The bilinear point-line connexion in space: an extension of Clebsch's connexion."

(22) MR. E. A. HOOK: "Multiple points on Lissajous's curves in two and three dimensions."

(23) DR. L. P. EISENHART: "Infinitesimal deformation of the skew helicoid."

(24) PROFESSOR JOHN EIESLAND: "Null systems in space of five dimensions."

(25) PROFESSOR E. D. ROE: "Note on a partial differential equation."

(26) DR. SAUL EPSTEIN: "On integrability by quadratures."

(27) MR. W. B. FORD: "On the possibility of differentiating term by term the developments for an arbitrary function of one real variable in terms of Bessel functions."

(28) PROFESSOR T. F. HOLGATE: "Apolar triads on a straight line and on a conic."

(29) PROFESSOR H. B. NEWSON: "List of continuous groups of collineations in space."

(30) MISS HELEN BREWSTER: "The group of collineations leaving a quadric surface invariant."

(31) PROFESSOR ALEXANDER PELL: "On the generalized Beltrami problem."

(32) PROFESSOR L. HEFFTER: "Ueber Curvenintegrale in m -dimensionalem Raum."

The papers of Dr. Dunkel and Mr. Hook were presented to the Society through Professor Bôcher, those of Professor Loewy and Professor Heffter through Professor Moore, that of Dr. Epsteen through Professor Lovett, and Miss Brewster's through Professor Newson. In the absence of the authors, Dr. Huntington's paper was read by Professor Dickson, Professor Westlund's and Professor Emch's by Professor Moore, Professor Loewy's by Professor Maschke, Dr. Snyder's and Professor Field's by Dr. Fite, and Miss Brewster's by Professor Newson. The papers of Dr. Collins, Dr. Dunkel, Dr. Keyser, Mr. Hook, Dr. Eisenhart, Professor Roe, Mr. Ford, and Professor Heffter were read by title.

Professor Bolza's paper was published in the October number of the BULLETIN. The papers of Dr. Fite and Dr. Epsteen will appear in later numbers of the BULLETIN. Abstracts of the other papers presented are given below. The abstracts are numbered to correspond to the titles in the list above.

1. In constructing expressions for the elements of the planetary orbits three things are desirable: (*a*) that they shall, if they are series, converge for at least a finite interval of time; (*b*) that it shall be possible practically to find their coefficients; and (*c*) that the time for which they are valid shall be very long. The method of developing the elements as power series in the masses can be proved to give results which converge for a finite interval of time, but these series have the practical inconvenience of containing secular terms even of the first order. The thought has been that these secular terms may be the expansions of periodic terms. Following out this idea, Lagrange has succeeded by quite arbitrary and up to the present unjustified processes in avoiding the secular terms of the first order in the masses and in the eccentricities. Secular terms of higher orders appear, and even if they did not, his results in no way prove the stability of the solar system, as is frequently stated. Dr. Moulton inquires whether these secular terms may not be avoided by processes which can be proved to be valid. The answer is in the affirmative, although the method of treatment differs from that of Lagrange. The series which are used fulfill condition (*a*); they fulfill (*b*) about as well as those of the standard methods; and they are probably valid for a much longer time than those now in use.

2. Professor Hathaway employs the quaternion analysis in the problem of three bodies, with the hope that the new and concise forms of that analysis may suggest important developments. Let α, β, γ be the vector sides BC, CA, AB of a fixed triangle of reference, and put

$$S \cdot \beta\gamma = 1/a, \quad S \cdot \gamma\alpha = 1/b, \quad S \cdot \alpha\beta = 1/c, \quad UV\alpha\beta = k.$$

Then if P, Q, R be the positions of masses a, b, c at any time t , there is a definite strain ϕ such that

$$\phi k = 0, \quad \phi ABC = PQR$$

whose differential equation is

$$(1) \quad \ddot{\phi} = \phi \left(\frac{bcaSa}{T^3\phi\alpha} + \frac{ca\beta S\beta}{T^3\phi\beta} + \frac{ab\gamma S\gamma}{T^3\phi\gamma} \right) = \phi\pi, \text{ say,}$$

or in conjugate form, $\ddot{\phi}' = \pi\phi'$. The area and energy integrals are

$$(2) \quad \dot{\phi}\phi' - \phi\dot{\phi}' = V\epsilon, \quad \frac{1}{2}S_1\dot{\phi}'\dot{\phi} + S_1\phi'\phi\pi = C,$$

where ϵ is the perpendicular to the invariant plane, and C is the sum of the kinetic and potential energies.* Forms are reduced when expressed in terms of $\psi = \phi'\phi$, and π is already reduced since $T^2\phi\rho = -S \cdot \rho\psi\rho$.

We have $\dot{\phi}'\phi - \phi'\dot{\phi} = xVk$, which gives by differentiation $\pi\psi - \psi\pi = \dot{x}Vk$, a reduced differential equation for x . Also by differentiating ψ twice we find $2\dot{\phi}'\dot{\phi} = \Phi$, say, in reduced form, so that the identity

$$[k(\dot{\psi} + xVk)k\psi k(\dot{\psi} - xVk) + 4S_2\dot{\psi} \cdot k\Phi]^2 = 0$$

or

$$(3) \quad x^4 + 2x^2(S \cdot k\psi k\Phi + S_2\dot{\psi}) + 4xS_1 \cdot k\psi k\psi k\Phi + S_2\dot{\psi}^2 + 4S_2 \cdot \psi\Phi - 2S_1 \cdot k\dot{\psi}k\psi k\dot{\psi}k\Phi = 0$$

is a reduced equation for x itself. This agrees with Lagrange, x being to a constant factor his ρ ; but does not agree with

* The cubic in any ϕ is $\phi^3 - S_1\phi \cdot \phi^2 + S_2\phi \cdot \phi - S_3\phi = 0$. Thus S_r is an operator on a strain which gives a characteristic scalar. Products under S_r are cyclically commutative and replaceable by their conjugates. S_1 is distributive over a sum, and S_2 over a product. In the present case $S_3 = 0$, and S_2 has the distributive property of S_3 for coplanar factors.

Dziobek [Planetary motions, page 56 (19)]. I have verified, directly and indirectly, that (3) is a particular integral of the differential equation for x in connection with (1), the general integral being found by replacing x by $x + C$ in (3). This settles a question which Dziobek says was undetermined (Planetary motions, page 57).

By differentiating ψ three times and substituting, we find the reduced form of (1)

$$(4) \quad \ddot{\psi} = 2(\dot{\psi}\pi + \pi\dot{\psi}) + \pi\dot{\psi} + \psi\pi + x(Vk\pi - \pi Vk).$$

This is equivalent to three ordinary equations of third order between the mutual distances of the bodies, found by operating with S_1 , $S_1 \cdot \psi$, $S_1 \cdot \pi$. The first two are straight differentials of the reduced forms of (2), so that with these integrations the final forms are

$$(5) \quad S_1 \ddot{\psi} + 2S_1 \cdot \psi \pi = 4C,$$

$$(6) \quad S_1 \cdot \psi \ddot{\psi} = 2S_1 \cdot \psi^2 \pi + \frac{1}{2} S_1 \cdot \dot{\psi}^2 - x^2 + 2T^2 \epsilon,$$

$$(7) \quad S_1 \cdot \pi \ddot{\psi} = 2S_1 \cdot \pi \dot{\psi} + 4S_1 \cdot \pi^2 \dot{\psi}.$$

Dziobek does not note that the area integral (6) is one of the integrals of (4).

3. The different branches of vector analysis are very much alike in several respects. They deal with the same fundamental concept, the vector, and employ largely the same derived concepts, the scalar and vector products and their combinations. Apparently then there is no reason why a single consistent notation should not be used in all the branches of the subject. Instead of this we seem to be tending in the opposite direction, as is evidenced in Professor Gibbs's new book.

Professor Gibbs has made certain improvements in the notation which will doubtless stand. He avoids Hamilton's cumbersome T and U and puts his signs of multiplication *between* instead of to one side of the factors, thus admitting of a new sign of operation being put on either side. This Professor Gibbs was led to do doubtless through his study of dyadics. However, instead of retaining Hamilton's letters or Grassmann's notation for the scalar and vector products, he has introduced two new marks, the dot and the cross, to denote them.

Now all interests can be easily harmonized in the following way:—Let a^*b , a^ab , a^ab , ab denote respectively Sab , Vab , quaternion ab , dyadic ab . Also let a^*b^*c be denoted by $[abc]$, as is done by Gibbs, whenever this notation is more convenient. In this way are retained Hamilton's notation, and the good points in Gibbs's and Grassmann's notations. There is another very good reason why Gibbs's dot and cross should be given up and the letters retained. It is that the use of letters admits of an easy extension of the notation to new products. That there are such besides the quaternion the author has demonstrated by constructing them and putting them to practical use.

4. In the usual theory of higher complex members, the coördinates are either real numbers or else ordinary complex quantities. To avoid the resulting double phraseology and to attain an evident generalization of the theory, Professor Dickson considers systems of complex numbers whose coördinates belong to an arbitrary field F . The usual definition rests upon a multiplication table for n units. In essence, a system of complex numbers depends upon n^3 constant marks γ_{ikl} of F subject to the three conditions

$$\sum_{i=1}^n \gamma_{ikl} \gamma_{jlt} = \sum_{j=1}^n \gamma_{klt} \gamma_{ijl} \quad (i, k, l, t = 1, 2, \dots, n),$$

$$\Delta_a \equiv \left| \sum_{i=1}^n \gamma_{ikl} a_i \right| \text{ not zero for every } a_1, \dots, a_n,$$

$$\Delta'_a \equiv \left| \sum_{k=1}^n \gamma_{ikl} a_k \right| \text{ not zero for every } a_1, \dots, a_n.$$

These conditions are shown to be independent by using systems with $n = 2$.

For the second definition the elements are $A = (a_1, a_2, \dots, a_n)$, where $a_1, \dots, a_n$ are marks of F . Addition is defined thus:

$$A + B = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n).$$

A second rule of combination of elements is defined by four properties: (1) For any two elements A and B of the system, $A \cdot B$ is an element whose coördinates are bilinear functions of the coördinates of A and B , with fixed coefficients belonging to F . (2) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$, if $A \cdot B$, etc., belong to the

system. (3) There exists in the system an element I such that $A \cdot I = A$ for every element A . (4) There exists in the system at least one element A such that $A \cdot Z \neq 0$ for any element $Z \neq 0$. The elements are shown to form a system of complex numbers according to the usual definition. The four properties are shown to be independent. The paper will appear in the *Transactions*.

5. Professor Dickson's second paper gives a definition of a field by means of nine independent postulates and a second definition by means of eleven independent postulates. Of various possible sets of postulates, these two sets (especially the second) appear to present most naturally the elementary properties necessarily holding for a field and in a manner most suitable for actually testing for the field property.

A set of elements with two rules of combination denoted by $\circ$ and $\square$ forms a field if the following properties hold: (1) If a and b belong to the set, then also $a \circ b$ belongs to the set. (2) $a \circ b = b \circ a$, whenever $a \circ b$ and $b \circ a$ belong to the set. (3) $(a \circ b) \circ c = a \circ (b \circ c)$, whenever $a \circ b$, $b \circ c$, $(a \circ b) \circ c$, and $a \circ (b \circ c)$ belong to the set. (4) For any two elements a and b of the set, there exists in the set an element x such that $(a \circ x) \circ b = b$. (5), (6), (7) are the same as (1), (2), (3) with $\circ$ replaced by $\square$. (8) For any two elements a and b of the set, such that $c \square a \neq a$ for at least one element c , there exists in the set an element x such that $(a \square x) \square b = b$. (9) $a \square (b \circ c) = (a \square b) \circ (a \square c)$, whenever $a \square b$, $a \square c$, etc., belong to the set.

The second set employs postulates (1), (2), (3), (5), (6), (7), (9) and the following four: (4') There exists in the set an element z such that $z \circ b = b$ for every element b . (4'') If elements z of character (4') exist, then for some such element z and for every element a , there exists an element x for which $a \circ x = z$. (8') There exists an element u such that $u \square b = b$ for every element b . (8'') If elements u of character (8') exist, then for some such element u and for every element a such that $c \square a \neq a$ for at least one element c , there exists an element x for which $a \square x = u$.

This paper will appear in the *Transactions*.

6. Dr. Huntington's paper presents several sets of postulates which define a field, that is, an assemblage in which all the

rational operations of algebra can be performed. The simplest of these sets contains the following seven postulates, where the symbols $\oplus$ and $\odot$ indicate abstract operations exemplified by addition and multiplication of rational numbers:

1) If a, b and either $a \oplus b$ or $b \oplus a$ belong to the field, then $a \oplus b = b \oplus a$.

2) If $a, b, c, a \oplus b, b \oplus c$ and either $(a \oplus b) \oplus c$ or $a \oplus (b \oplus c)$ belong to the field, then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

3) For every two elements a and b ($a = b$ or $a \neq b$) there is an element x in the field such that $a \oplus x = b$.

4), 5) The same as 1) and 2), when $\oplus$ is replaced by $\odot$.

6) For every two elements a and b ($a = b$ or $a \neq b$), provided $a \oplus a \neq a$, there is an element y in the field such that $a \odot y = b$.

7) If $a, b, c, b \oplus c, a \odot b, a \odot c$ and either $a \odot (b \oplus c)$ or $(a \odot b) \oplus (a \odot c)$ belong to the field, then $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

The independence of the postulates of each set is established by the now familiar method of Peano and Hilbert. The paper will appear in the *Transactions*.

7. In this paper, which will be published in the *Proceedings the American Academy of Arts and Sciences*, Mr. Dunkel considers a system of differential equations

$$\frac{dy_i}{dx} = \sum_{j=1}^{i=n} \left(\frac{\mu_{i,j}}{x} + a_{i,j} \right) y_j \quad (i = 1, 2, \dots, n)$$

in which the $\mu_{i,j}$'s are constants and the $a_{i,j}$'s are functions of the real independent variable x , not necessarily analytic, but continuous in the interval $0 < x \leq b$. The absolute values, $|a_{i,j}|$, multiplied by certain positive integral powers of $\log x$ are required to be integrable up to the point $x = 0$; these powers vary according to the set of solutions being developed, and they may in some cases be zero. When the $a_{i,j}$'s satisfy these conditions, the point $x = 0$ is called a regular singular point. Use is made of a theorem on elementary divisors to reduce the system of equations to a canonical form. Solutions of the canonical system are developed by the method of successive approximations; and from these the form of the solutions of the original system is then determined. The results thus obtained are applied to the single differential equation

$$\frac{d^n y}{dx^n} + \left(\frac{\mu_1}{x} + p_1\right) \frac{d^{n-1} y}{dx^{n-1}} + \left(\frac{\mu_2}{x} + p_2\right) \frac{1}{x} \frac{d^{n-2} y}{dx^{n-2}} + \dots$$

$$+ \left(\frac{\mu_n}{x} + p_n\right) \frac{1}{x^{n-1}} y = 0,$$

in which the μ 's are constants, and the p 's are functions of the real independent variable x satisfying the same requirements that were imposed upon the functions a_{ij} .

Let r_κ be a root of multiplicity e_κ of the indicial equation. The multiplicities of all the roots of this equation whose real parts are the same as that of r_κ are considered, and e_κ is chosen as any one of these multiplicities as great as any other in this set. It is then required that the absolute values of the p 's after being multiplied by $(\log x)^{e_\kappa-1}$ be integrable up to $x=0$; and with this requirement it is found that we can develop the following e_κ solutions and their first $n-1$ derivatives corresponding to the root r_κ :

$$y^{\kappa\lambda} = x^{r_\kappa} (\log x)^{e_\kappa-\lambda} E_{n-\lambda}^{\kappa\lambda},$$

$$(\lambda = 1, 2, \dots, e_\kappa),$$

$$\frac{d^i y^{\kappa\lambda}}{dx^i} = x^{r_\kappa-i} (\log x)^{e_\kappa-\lambda} E_{n-i}^{\kappa\lambda},$$

where the functions $E_{n-i}^{\kappa\lambda}$ are continuous in an interval $0 \leq x \leq c$, and

$$E_n^{\kappa\lambda}|_{x=0} = 1, \quad E_{n-i}^{\kappa\lambda}|_{x=0} = r_\kappa(r_\kappa - 1) \dots (r_\kappa - i + 1).$$

In the case $n=2$ these results not only yield the theorems which were established by Professor Bôcher in the *Transactions* for January, 1900, but for the case of equal exponents go slightly further.

9. Professor Shaw's paper gives the reduction of all numbers of linear associative algebras to typical forms, and a new basis of classification of such algebras. It makes use of a previously discovered form involving units of the type λ_{ijk} such

that

$$\lambda_{ijk} \cdot \lambda_{i'j'k'} = 0 \quad \text{if } j \neq i'$$

$$= \lambda_{i'j'k+k'} \quad \text{if } j = i'.$$

The units are also subject to restrictions as to the size of the num-

ber k , represented by the third subscript. The method is given of deriving all algebras to which belong numbers of which the most general number satisfies a given characteristic equation. The place of Peirce's, Scheffer's and Molien's investigations is also shown, with an examination of the boundaries of these investigations and an extension and generalization of their results. The paper is a conclusion to one presented before the Chicago Section, January, 1902.

11. The moduli of periodicity for the normal integrals of the first kind on the Riemann surface for $w^4 = (z - \kappa_1)^2(z - \kappa_2)^2(z - \kappa_3)(z - \kappa_4)^3$ depend on a single parameter ω which undergoes a linear substitution by every monodromy of the branch points. Mr. Young's paper considers the group Γ of all such substitutions, which is found to be generated by the two, $\omega' = \omega + \sqrt{2}$ and $\omega'' = -1/\omega$. All functions automorphic for the group are shown to be rationally expressible in terms of the anharmonic ratio of the branch points.

The properties of Γ are studied in detail, the investigation following the lines laid down by Klein in his treatment of the (elliptic) modular functions, and results analogous to practically all of Klein's results are obtained. The most noteworthy difference is found in the fact that Γ contains an infinite series of subgroups of genus $p = 0$. These subgroups are of index $2n$ and are isomorphic with the dihedral group of order $2n$; two sets of generators are determined for each.

The paper closes with a discussion of the quotient groups of Γ with the "principal" congruence subgroups, the latter consisting of all substitutions of Γ that are congruent to identity (mod. q). When q is an odd prime, the quotient group is shown to be of order $\frac{1}{2}q(q^2 - 1)$ or $q(q^2 - 1)$ according as q is of the form $8h \pm 1$ or $8h \pm 3$. In the former case the quotient group is simply isomorphic with Klein's $G_{\frac{1}{2}q(q^2-1)}$; in the latter case it contains a subgroup of half its order which is simply isomorphic with $G_{\frac{1}{2}q(q^2-1)}$.

12. Dr. Westlund's paper, which is an extension of a paper presented at the meeting of the Chicago section in January last, treats of the relation between the class numbers of the cyclotomic number field $k\left(e^{\frac{2\pi i}{p^n}}\right)$, when p is any odd prime, and its subfield $k\left(e^{\frac{2\pi i}{p^{n-1}}}\right)$. The method used is similar to that used by Weber for the case $p = 2$.

13. Professor Dickson's third paper is an addition to his article, "Theory of linear groups in an arbitrary field," *Transactions*, volume 2 (1901), pages 363-394. By certain modifications, the methods of §§ 9-10 may be made to apply to the there excluded case of modulus 2, provided $n > 1$. Hence there results a new system of simple groups of order $2^{6n}(2^{6n}-1)(2^{2n}-1)$, for any integer $n > 1$. There remains for subsequent investigation the case $n = 1$, the order being then 12096. For $n = 2$ and $n = 3$ the orders of the new simple groups are respectively $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$, $2^{18} \cdot 3^5 \cdot 7^2 \cdot 19 \cdot 73$.

14. The paper of Professor Loewy concerns the reducibility of groups of linear homogeneous substitutions, and is in abstract as follows: A group G of linear homogeneous substitutions in n variables is styled reducible if there exist $m < n$ linear homogeneous functions of the variables with constant coefficients, such that under the group they are transformed in a linear homogeneous way. The paper considers the following fundamental theorem: To every reducible group G of linear homogeneous substitutions there belongs a definite finite number of irreducible groups of linear homogeneous substitutions, which are uniquely determined apart from the order, in case one considers similar groups as identical. This theorem is applied in the first place to the theory of groups of a finite number of linear homogeneous substitutions and then to the theory of those infinite groups introduced by the author in volume 53, page 225, of the *Mathematische Annalen* and designated as groups of the type of a finite group. Further, there is introduced the more general notion of reducibility or irreducibility of a group G of linear homogeneous substitutions with respect to a realm of rationality or field (Körper) Ω to which all the coefficients of the various substitutions of the group belong, and the fundamental theorem is treated from this standpoint. In conclusion, the following theorem is proved: In order to exhibit all the finite groups belonging to a given field Ω , it is sufficient to exhibit all the finite groups which are irreducible with respect to the field Ω .

15. A twisted cubic has three osculating planes through every point. There are always two points in which these three planes are perpendicular to one another. If there are more than two such points, there must be an infinite number,

and their locus is then called the directrix of such a specialized cubic curve. For the case where the curve is a parabola, *i. e.*, if the plane at infinity osculates the curve, O. Böklen and W. F. Meyer have shown that its directrix is a straight line, and have discussed the conditions necessary for its existence. Professor White's paper is an examination of the same problem for cubical ellipses and hyperbolas. By analogy with plane conics one would look for a curvilinear directrix, if any; but it is proven that it must be straight. The conditions for its existence are found in the identical vanishing of a ternary quadric, giving three conditions of which, however, only two are independent. Meyer's twisted cubics with a directrix are defined by three conditions, while this new class is subject to only two; the latter, however, does not include the former class, but the two together comprise all possible twisted cubics with directrix.

16. Professor Morley's memoir is a continuation of one published in the *Transactions*, volume 1, page 97. What corresponds, for a plane n -line, to the elementary facts that for a 3-line the perpendiculars meet at a point, and for a 4-line the 4 such points lie on a line?

First, four lines determine uniquely a hypocycloid of class 3, Δ^3 ; if from the center of the Δ^3 touching each 4 of 5 lines a perpendicular be drawn to the fifth line, these five perpendiculars meet at a point; for 6 lines the 6 such points lie on a line. So there is a quite simple curve Δ^{2n-1} uniquely determined by $2n$ lines; if from the center of the Δ^{2n-1} touching each $2n$ of $2n + 1$ lines a perpendicular be drawn to the line omitted, these perpendiculars meet at a point and for $2n + 2$ lines the $2n + 2$ such points lie on a line. The curve Δ is for metrical purposes very convenient, much more so than Clifford's n -fold parabola, also uniquely determined by $2n$ lines. Second, if from the nine-point center, or center of Feuerbach's circle, of each 3 of 4 lines, a perpendicular be drawn on the line left out, these perpendiculars meet at a point. And the center of a circle arising from n lines has the analogous property. Thus we have for an even number of lines an orthocenter and a directrix; for an odd number, two and therefore a line of orthocenters.

17. Dr. Snyder has applied the same methods which he employed in the study of unicursal and elliptic scrolls of order six to obtain the forms and equations of those of genus greater

than 1. There are eleven types of genus 2, five of genus 3 and two of genus 4. The scrolls cannot have a triple conic nor more than one double conic as part of the nodal curve. Added to the paper is a scheme showing that the maximum genus of a scroll of order $2n$ is $(n-1)^2$, and of a scroll of order $2n-1$ is $n(n-1)$.

18. Professor Field obtains the general equation of a quintic curve having five fixed double points. It follows from the form of this equation that if a conic and cubic be taken and breaks made at two of their points of intersection, there result quintic curves. The curves are classified according to the sequence of the double points. Forty-six forms are obtained.

19. Dr. Keyser's paper is devoted primarily to a systematic introduction to the line and plane geometries of four-space, and secondarily to a correlation of these with kindred doctrines, such as the circle geometry of ordinary space and the theory of the four-space point conceived as a plenum of circular cones. The point and lineoid (three-space) theories of four-space, in so far as needed, are assumed. By reason of the locus and envelope phases of the line and plane, one has to do with four fundamental element aspects. These are strictly four distinct elements, giving rise to four distinct geometric theories which, like the elements, fall into reciprocal pairs. In a first chapter, by means of an arbitrary finite pentaedron of reference, *i. e.*, the configuration determined by any five non-collineoidal points or any five non-copunctal lineoids, four appropriate homogeneous coördinate systems are established, and their relationships disclosed in connection with the homographic and dualistic transformations. Each of these systems is composed of ten variables whose nine ratios, being connected by five (equivalent in general to three independent) quadratic identities, reduce to the necessary and sufficient number six of independent functions. These identities, being essentially like those which appear in the circle theory of ordinary space, are of order five, whence it appears that four arbitrary lines (planes) determine uniquely a fifth and so give rise to a configuration of lines (planes) analogous to the "pentacycle" of Stephanos. A second chapter which is concerned with the solution of preliminary elementary problems in terms of the adopted coördinate systems, is followed by a third devoted to a somewhat detailed exposition of

the above-mentioned correlation. The sphere (of ordinary space) may be conceived as an orthosphere, *i. e.*, a sphere regarded as the assemblage of spheres orthogonal to it; or it may be regarded as the envelope of all the orthospheres containing it. Calling the sphere an orthosphere or a sphere according as it is conceived in the former or in the latter way, and pairing the spheres and orthospheres (of ordinary space) respectively with the points and lineoids of four-space, it is seen that to the union of point and lineoid corresponds orthogonality of sphere and orthosphere — a principle which facilitates the parallelization of a theory of either space with its correspondent in the other. Among such readily deducible correspondences may be mentioned: the circle (and its geometry) has four aspects matching the four element aspects (and corresponding theory phases) mentioned above; the geometry of circles orthogonal to a sphere is analytically identical with the line (plane) geometry of a lineoid (four-space point); the theory of circles in involution with a given circle is parallel to the theory of lines (planes) having a point (or lineoid) in common with a given plane (line); and the geometry of circles in bi-involution with a given circle corresponds proposition for proposition with the four-space geometry of coplanar lines or of collinear planes.

20. In a previous paper, read before the society (Chicago) June 1, 1902, and to be published in the *Transactions*, Profes-

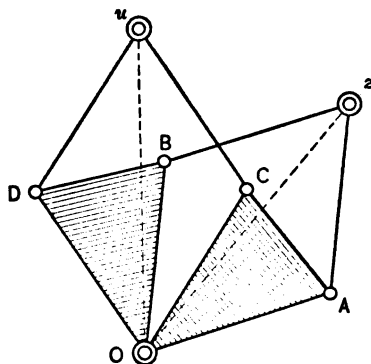


FIG. 1.

sor Emch has shown that all algebraic transformations of complex variables may be realized by linkages. In this paper he describes a very simple linkage realizing the rotation $u = e^{i\theta} \cdot z$,

where θ is constant. In Fig. 1, all links with the exception of AC and BD are equal. If $AC = BD$ are chosen in such a manner that

$$\angle AOC = \angle BOD = \theta,$$

then it can easily be proved that $\angle zOu = \theta$, no matter how the linkage may be distorted. Thus, if z is moved into any part of the complex plane, it is clear that the point u represents the complex number $u = e^{i\theta} \cdot z$.

Taking $\theta = 2\pi/n$, $n - 2$ equal cells may be successively combined in exactly the same manner, as $OCuD$ is attached to $OAzB$, thus forming a closed linkage of n cells. The vertices $z, u, u_1, \dots$ of this linkage form a regular variable polygon of n sides. If

$$z = \sqrt[n]{w} = r^{1/n} e^{i(\phi + 2k\pi)/n} \quad (k = 0, 1, 2, \dots, n-1; w = re^{i\phi}),$$

then the other vertices of the polygon represent the remaining roots. This compound linkage has been devised to show in a simple case the realization of many valued functions by linkages, which in general presents considerable difficulties.

21. The plane connex of Clebsch is defined by equating to zero a double ternary form containing a set of point coördinates and a set of line coördinates. The analogous space connex, defined by forms involving the coördinates of a point and of a plane, has been studied by Krause (*Mathematische Annalen*, 1879), Sintsof and others. The paper of Dr. Kasner gives another extension to space by introducing forms involving a set of point coördinates and a set of line coördinates. The vanishing of such a quaternary-senary form $a_x \cdot A_P^m$, where x or $x_1 : x_2 : x_3 : x_4$ represents the point and P or $P_{12} : P_{13} : P_{14} : P_{34} : P_{42} : P_{23}$ the line, defines the general *point-line connex* of space. After a survey of the general case, of the connection with certain types of differential equations, and of the configurations defined by the intersection of two or more connexes, the author considers in detail the case where each set of variables is involved linearly, $n = 1, m = 1$.

To each point x there corresponds, by means of this bilinear connex, a linear line complex, and to each line P there corresponds a plane. The study of the connex consists in the discussion of these two transformations, which are both linear.

To enumerate a few of the results: The locus of the point whose corresponding linear complex is special is a quadric surface. There are five points, situated on this quadric, each possessing the property that the corresponding complex is special and has for its directrix a line passing through the point. There are two fundamental lines $\bar{P}$, i. e., such that the corresponding plane is indeterminate. The connexion is defined geometrically by means of the (1, 1) correspondence which may be set up between the lines of a linear congruence and the points of a quadric surface. The paper concludes with the discussion of certain invariants and covariants and the establishment of normal forms with respect to both cogredient and digredient transformations.

22. Mr. Hook's paper, which will be published in the *Annals of Mathematics*, deals with curves given by either two or three equations of the form

$$x_i = \cos m_i(t + t_i) \quad (i = 1, 2 \text{ or } 1, 2, 3),$$

m_i and t_i being constants, x_i the cartesian coördinate, and t a variable parameter. The plane curves are classified into open and closed curves. The three-dimensional curves fall into four types based on the number of open or closed curve projections they possess on the three coördinate planes. This classification applies only to the case in which the ratios of the quantities m_i are commensurable. Transcendental curves are not considered at all. For each type of plane curves a formula is found for the number of real double points. On the space curves both double and quadruple points are possible and formulæ giving the numbers of these on each type of curve are found. A number of interesting theorems in regard to the relative positions of the multiple points on the curves are established. It is found that in all cases the tangents at the multiple points are real and distinct.

23. Dr. Eisenhart has applied the methods of Weingarten to the study of the infinitesimal deformation of the skew helicoid and has found that the solution is complete, for the characteristic partial differential equation of the second order can be integrated. The cartesian coördinates of the characteristic surfaces, which correspond to the helicoid with orthogonality of linear elements, are given by quadratures. They constitute a

general class of surfaces of which the surfaces of revolution form a subclass, corresponding to a constant value of one of the two arbitrary functions which enter into the general integral of the characteristic equation. When these surfaces are surfaces of revolution, and only in this case, their lines of curvature correspond to the asymptotic lines on the helicoid.

The associate surfaces of the deformation are given without quadrature and are found to be the general moulded surfaces of Monge. Moreover, when any moulded surface is given, the characteristic function which determines the corresponding deformation can be found at once. Surfaces of revolution are moulded surfaces and it is shown that when the associate surface is a surface of revolution the characteristic surface also is a surface of revolution, and conversely.

24. Professor Eiesland's paper contains a study of the null system

$$dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0,$$

all the lines of which may by means of Lie's transformation

$$x_1 = \frac{1}{2} P_1, \quad x_2 = X, \quad x_3 = \frac{1}{2} P_2, \quad x_4 = X, \quad x_5 + x_1 x_2 + x_3 x_4 = X_5$$

be transformed into ∞^5 parabolas lying in planes parallel to the X_5 -axis. The relations between the ∞^7 lines of the null system and these ∞^5 parabolas have been discussed. As an interesting application of this theory, a class of surfaces in ordinary space was found whose asymptotic lines are known. In the case of certain two-dimensional translation surfaces in the space M_5 , a class of surfaces in ordinary space was obtained which Darboux arrived at in his *Leçons*, volume I, page 141, by an entirely different method.

In the last part of the paper the question of invariance of the null system in n -dimensional space is discussed; of all the euclidean motions $\left(\frac{n-1}{2}\right)^2$ rotations and a translation along an axis transform all the lines of the null system into themselves. For $n = 5$ the following theorems have been proved: 1° There exist ∞^{21} projective transformations which leave the null system invariant. 2° There exist ∞^{21} contact transformations leaving the differential equations

$$\frac{d^3 X_3}{dx_1^3} = 0, \quad \frac{d^2 X_2}{dx_1^2} = 0$$

invariant. 3° There exist ∞^{10} projective transformations which leave invariant the complex of lines defined by the differential equations

$$dx_6 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0,$$

$$dx_1 dx_2 + dx_3 dx_4 = 0.$$

This complex has been called an asymptotic complex on account of the rôle it plays in the transformation of surfaces in M_5 into surfaces of ordinary space on which the asymptotic lines are known; in fact, the lines of this complex have been shown, when transformed by Lie's transformation to become linear tangents along the asymptotic curves of the surface. 4° There exist only one rotation and a translation along the X_5 -axis leaving an asymptotic complex invariant.

25. In Professor Roe's paper an elementary proof is given that any function $\phi(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$ which satisfies the partial differential equation

$$\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + \dots + y_n \frac{\partial}{\partial x_n} \right) \phi = 0,$$

where the y 's are independent of the x 's, is a function of any set of $n - 1$ independent determinants of the x 's and y 's of the form

$$(xy) = x_i y_{i+1} - x_{i+1} y_i.$$

The paper will be published in the *Annals of Mathematics*.

27. Mr. Ford's paper is concerned with the determination of sufficient conditions for a function $f(x)$ of the real variable x in order that each of the two following developments, occurring in mathematical physics, for $f(x)$ in terms of Bessel's function $J_\nu(x)$ may be differentiated term by term:

$$\text{I.} \quad f(x) = 2 \sum_1^\infty \frac{J_\nu(\lambda_n x)}{J_\nu'^2(\lambda_n)} \int_0^1 x f(x) J_\nu(\lambda_n x) dx,$$

where λ_n is one of the positive roots of the equation $J_\nu(x) = 0$;

$$\text{II.} \quad f(x) = (2\nu + 2) \int_0^1 f(x) x^{\nu+1} dx \\ + 2 \sum_1^\infty \frac{J_\nu(\lambda'_n x)}{J_\nu'^2(\lambda'_n)} \int_0^1 x f(x) J_\nu(\lambda'_n x) dx,$$

where λ_n is one of the positive roots of the equation $xJ_n'(x) - vJ_n(x) = 0$.

The results obtained are summarized in the following :

Theorem : Each of the series I and II converges when $a' < x < b'$ ($0 < a' < b' < 1$) to the limit $f(x)$, and when differentiated term by term the resulting series converge for the same values of x to the limit $f'(x)$, provided that $v > -\frac{1}{2}$ and that the function $\phi(x) = x^{-v} f(x)$ satisfies the following conditions :

Condition A : $\phi(x)$ is continuous, or is made up of a finite number of a continuous portions throughout the interval $0 \leq x \leq 1$.

Condition B : $\phi(x)$ possesses a derivative $\phi'(x)$ which is continuous throughout the interval $a' \leq x \leq b'$, and in the intervals $0 \leq x < a'$, $b' < x \leq 1$ is either continuous or is made up of a finite number of continuous portions. Also, the function $\frac{\phi'(x)}{x}$ is finite in the neighborhood of the point $x = 0$.

Condition C : $\phi(x)$ possesses a second derivative $\phi''(x)$ which is finite throughout the interval $a' - \epsilon \leq x \leq b' + \epsilon$ (ϵ arbitrarily small and positive).

Condition D : $\phi(1) = 0$.

Moreover, when $-1 < v \leq -\frac{1}{2}$ the above theorem is true if we require also that the function $|x^v f(x)|$ be adapted to integration in the neighborhood at the right of the point $x = 0$.

28. Defining two homogeneous binary forms $f(x, y)$ and $\phi(x, y)$, each of the n th order, as apolar when they are so related that the invariant $\left(\frac{\partial f}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial \phi}{\partial x} \right)^n$ vanishes identically, Professor Holgate finds a linear construction of the third element in one of two apolar cubic forms when its two remaining elements and the three elements of the other form are given. If A_1, A_2, A_3 are the three root points of $f^3(x, y) = 0$ and B_1, B_2, B_3 the three root points of $\phi^3(x, y) = 0$, then if B''' is the harmonic conjugate of B_1 with respect to A_1 and A_2 and C''' the conjugate of A_3 in the involution determined by the pairs A_1, A_2 ; B_1, B''' ; similarly if B' and B'' are the harmonic conjugates of B_1 relative to the pairs A_2, A_3 and A_3, A_1 , respectively, and C' and C'' the conjugates of A_1 and A_2 in the involutions A_2, A_3 ; B_1, B' , and A_3, A_1 ; B_1, B'' , respectively; then $B', C'; B'', C''; B''', C'''$ is an involution such that any pair together with B_1 constitutes a triad apolar to A_1, A_2, A_3 . Hence B_2 and B_3 need only be a pair in this involution, and if either of

these points is given the other can be readily found by linear constructions.

29. In the theory of continuous groups of transformations the collineation groups are of prime importance. In Lie's works are to be found complete lists of continuous groups of collineations in the plane, but nowhere does he give a list of all such groups in space of three dimensions. Professor Newson's paper contains a list of 490 continuous groups of collineations in space; this list is thought to be complete. A synthetic method was employed in determining these groups which is wholly independent of Lie's methods. These 490 groups are classified according to the thirteen well-known types of collineations in space and each is designated by a convenient symbol. This table is a natural extension of the table of groups of collineations in the plane given in Professor Newson's memoir in the *American Journal of Mathematics*, Volume 24, Number 2.

30. The group of collineations leaving a quadric surface F invariant is one of the most important subgroups of the general projective group. Miss Brewster's paper contains an exhaustive study of the ∞^6 collineations in the group $G_6(F)$. These are included under the types I, III, VI, IX, X and XI (Professor Newson's classification) and one special collineation of type XII. These collineations combine to form twenty-five continuous groups and seven mixed groups. These groups are classified according to their types, their invariant figures determined and the structure of each group discussed with reference to continuous groups of other types and singular transformations. A table is given at the end showing each group, its symbol, its invariant figure and the corresponding group in Lie's table on pages 251-254 of the third volume of "Theorie der Transformationsgruppen."

31. Dr. Pell applied a method given by Darboux (Leçons, etc., volume I, page 53 ff.) to determine the surfaces whose geodesics can be represented upon the plane as two-parameter curves of the form

$$f_1(x, y) + \mu f_2(x, y) + \nu f_3(x, y) = 0.$$

The problem has been treated by Beltrami's method by Dr. Stecker in the *Transactions* of 1901-1902.

32. The paper of Professor Heffter, which will appear in the *Transactions*, extends to space of m dimensions considerations concerning curvilinear integrals previously developed by him in the *Göttinger Nachrichten* of February 8, 1902, for the case $m = 2$ of the plane. As the curve C of integration is taken the general rectifiable curve; in case the curve C lies within a closed continuous region of continuity of the integrand function, the usual definition of the integral receives a certain modification, in that the range of values of the limitand sum is enlarged; this modification is seen to simplify the development of the theory as regards the so-called fundamental theorem of the integral calculus, the theory of the curvilinear integral as function of the upper limit in case it is independent of the path C , and the theorem of Cauchy.

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Assistant Secretary.

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THE MEETING OF SECTION A OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, PITTSBURGH, PA., JUNE 28-JULY 3, 1902.

BOTH in attendance and in the number and character of the papers presented, the Pittsburgh meeting of Section A of the American Association for the Advancement of Science, was very successful. The Secretary has not preserved a memorandum of the members who were present, but he estimates that forty or more attended the different sessions. The Section met on Monday afternoon, June 30, the Vice-President, Professor G. W. Hough, of Northwestern University, presiding, to hear the address of the retiring Vice-President, Professor James McMahon, of Cornell University. In the absence of Professor McMahon, his address, the subject of which was "Some recent applications of function theory to physical problems," was read by Professor R. S. Woodward, of Columbia University. This address was published in *Science* for July 25.

Amongst the papers on the programme of the Section were three important reports: I, "A report on quaternions," by Professor Alexander Macfarlane, which will be published in the *Proceedings* of the Association. II, "A second report on

recent progress in the theory of groups of finite order," by Professor G. A. Miller, Stanford University, which was presented in abstract by Dr. W. B. Fite, Cornell, in Professor Miller's absence. This report appears in the present number of the BULLETIN. III, "A report on the theory of collineations," by Professor H. B. Newson, the University of Kansas, which was read by title, in Professor Newson's absence. This report will be published in the *Proceedings* of the Association. Titles and abstracts of the other papers follow below.

Professor David P. Todd, Amherst College Observatory, presented two papers :

I. "On the adaptability of the glycerine clock to the diurnal motion of astronomical instruments, particularly those used in photographing solar eclipses."

The glycerine clock is an accurately constructed cylinder, about four inches in diameter, in which travels a piston, the rate of flow of the glycerine being controlled by a series of needle valves. By attaching a mirror or objective to a frame, equatorially mounted, and setting the clock under one of its arms at any convenient distance from the axis, the requisite rate for counteracting the diurnal motion of the sun can be given by means of the needle valves. This permits very heavy weights to be thrown on the piston and thereby the vibration of the instrument by wind can be precluded. When the run of the piston is finished, the glycerine is pumped out of the top of the cylinder and forced back into the bottom, and the run is commenced over again.

II. "On a convenient type of finder for very large equatorials."

In equatorials above twenty inches in aperture, the ordinary finder is necessarily mounted so far away from the axis of the great telescope that its use occasions much inconvenience, on account of the distance of its eye piece from that of the great tube. To obviate this difficulty Professor Todd proposes to construct the finder with a pair of reflectors, either planes or prisms, set at 45° , and to mount its tube in rings or bearings. By turning the tube in these, the finder's eye piece can be brought as near the eye piece of the great tube as is desired, or pushed away from it to admit attachment or adjustment of subsidiary apparatus.

"Series whose product is absolutely convergent," Professor Florian Cajori, Colorado College.

This paper is a continuation of the subject as developed by the author in his previous papers (*Transactions of the American Mathematical Society*, volume 2, pages 25–36 (1901); *Science*, volume 14, page 395 (1901), *BULLETIN*, volume 8, pages 231–236 (1902), and in the article of Alfred Pringsheim (*Transactions of the American Mathematical Society*, volume 2, pages 404–412 (1901). Some of the results previously obtained, relating to absolutely convergent products of two or more series, are generalized and the method of treatment is simplified. The construction of pairs of divergent series with real or complex terms is given, such that the product of the two series is not only absolutely convergent, but equal to any desired value, including zero.

Professor George Bruce Halsted, University of Texas, presented two papers :

I. "A new treatment of volume."

By a sect calculus, the product of two sects is fixed. The area of a triangle is defined as half the product of base and altitude, which is proved independently of the side chosen as base. Then the volume of a tetrahedron is defined as one third the product of base and altitude, which is proved independently of the side chosen as base. Then is proven the theorem : The volume of any tetrahedron is equal to the sum of the volumes of all the tetrahedra which arise from the first by making successively a set of transversal partitions. Then the theorem : However a tetrahedron is cut by a plane, this partition can be obtained in a set of transversal partitions using not more than two other planes. Then is proven as a theorem : If a tetrahedron T is in any way cut by any planes into a certain finite number of tetrahedra T_n , then is always the volume of the tetrahedron T equal to the sum of the volumes of all the tetrahedra T_n . Then the volume of any polyhedron is defined as the sum of the volumes of any set of tetrahedra into which it is cut, and this is proved independently of the set taken. A corollary is that if a polyhedron be cut into polyhedra, the sum of their volumes equals its volume. Then a prismatoid is defined, and by cutting it into tetrahedra its volume is proved to be

$$V = \frac{a}{4} (B + 3S) \text{ [see Halsted's Elements].}$$

All the forms of elementary geometry follow as special cases

of this. For example, the volume of the cube with edge 1 is thus *proved* to be 1. Thus volume is completely treated without ratio, without continuity, without limits, without infinity or any infinite process.

II. "A new founding of spherics."

This founds two-dimensional spherics and deduces all its theorems by assuming three sets of postulates. It makes no use of euclidean space, no use of straight line, center, radius, no use of any theorem of solid geometry. Thus it is an establishing of two-dimensional Riemannian geometry. The second group of assumptions, the "betweenness" assumptions, are entirely new. There is no assumption that the shortest path between two points is on a straightest line or great circle. It is rigorously proved that two sides of a spherical triangle are together greater than the third. Thus is eliminated the objectionable assumption analogous to "the straight line is the shortest distance between two points."

"A new solar attachment," Professor Herbert A. Howe, Chamberlin Observatory, University Park, Colorado.

The purpose of this paper is to call the attention of teachers of astronomy to a new solar attachment, the "Shattuck solar," which is capable of illustrating the uses of a number of astronomical instruments. When it is attached to an ordinary engineer's transit, a teacher may give his students a good deal of practice with it in some of the fundamental processes of practical astronomy, without having recourse to expensive astronomical instruments. The instruments illustrated are the prism transit in the meridian, the zenith telescope, the prime vertical transit, the sextant, and the equatorial coudé.

"On the periodic solutions of the problem of three bodies," Professor E. O. Lovett, Princeton University.

Lagrange found five exact solutions of the problem of three bodies in each of which the bodies preserve an unvarying configuration which revolves with a uniform velocity. When the third body is of infinitesimal mass compared with the other two, it can describe small periodic orbits in the vicinity of the points where exact solutions exist. The latter points were called centers of libration by Gylden, and Darwin calls the infinitely small body an oscillating satellite. Hill pointed out the fertility of the notion and made a splendid application of it

in his lunar theory. Poincaré elaborated the mathematical theory in his celebrated researches, and we owe to Darwin an extended collection of examples of periodic orbits.

One of the most recent investigations of such orbits is a suggestive paper by Charlier, in No. 18 of the *Meddelanden från Lunds Astronomiska Observatorium*. In the *Monthly Notices of the Royal Astronomical Society* for November, 1901, Plummer has discussed some of Charlier's results in a more general manner.

It is the object of Professor Lovett's paper to determine the imaginary centers of libration and their corresponding orbits, and thus complete the analytical solution proposed by Charlier. The results cannot be expected to fit the sky, but they may be of some interest to mathematical astronomers. It appears that there are real periodic orbits corresponding to imaginary centers of libration.

"The rate of the Riefler sidereal clock, No. 56," Professor Charles S. Howe, Case School of Applied Sciences.

In order to obtain the best rate from a clock, it is necessary to have it under constant pressure and constant temperature. Constant temperature may be obtained in a clock room. Dr. Riefler, of Munich, is the only manufacturer in the world who makes a clock in an air-tight glass case. There are three of these clocks in the United States, one of which is at the Case Observatory. The mean daily rate of this clock for a trifle over three months was .116 of a second. The average daily variation from this mean daily rate was .015 of a second, and the maximum variation was .022 of a second. It is believed that these rates are the best ever published.

"A representation of the coördinates of the moon in power series which are proved to converge for a finite interval of time," Dr. F. R. Moulton, University of Chicago.

In the construction of a lunar theory three conditions are very desirable, if not absolutely essential, viz.: (a) that every step and the general process shall be proved to be logically valid, at least under certain conditions which are fulfilled by the moon; (b) that the labor of constructing the series shall not be so great as to be impracticable, and (c) that the arbitrary constants shall enter in such a way that they may be determined from observations with a high degree of precision.

The first condition has not been insisted upon heretofore. The nearest approach is in Delaunay's theory in which every one of the finite number of steps taken is separately valid, but in which the infinite sequence which is involved has not been shown to converge. Dr. Moulton imposes everywhere condition (a), and also (b) and (c) so far as possible.

By making use of Cauchy's existence theorem for the solutions of total differential equations and Mittag-Leffler's recent splendid researches on the representation of analytic function in extended areas by generalizations of power series, and by defining conveniently a certain anchor ring, Dr. Moulton arrives at a method of obtaining expressions for the coördinates of the moon which are valid for any length of time chosen in advance; and, moreover, it is possible to determine in advance a number of terms which will give an arbitrary degree of accuracy. These very general results fulfill condition (a), but unfortunately do not fulfill (b) and (c).

Another method is given depending upon the expansion of the expressions for the coördinates as power series in certain appropriately chosen parameters, the coefficients in the case of the linear variables being trigonometric functions of the time, and in the case of the angular variables, linear and trigonometric functions of the time. It is proved that these series converge for all values of the time not too remote from the initial epoch. When a sufficient number of terms have been taken so that the error committed shall be less than an arbitrary amount they may be rearranged as a Fourier series so that the expressions will be of the form given in the standard methods. This method fulfills all three of the conditions, but the second not so well as that first developed by Hill in his remarkable researches.

"The mass of the rings of Saturn," Professor Asaph Hall, South Norfolk, Conn.

The mass of these rings was first determined by Bessel in 1831 from the motion of the apsides of the orbit of Titan. This motion is about half a degree in a year. But the action of the figure of the planet, and the attractions of the other satellites were neglected; and, as Bessel pointed out, the resulting mass of the rings was too great. This mass is $1/118$, the mass of Saturn being taken as the unit.

In this paper an equation was formed containing two indeter-

minate quantities depending on the figure of the planet, the mass of the rings, and the masses of the three brighter satellites, Rhea, Dione and Tethys. The small resulting action of the other satellites was estimated. The coefficients of these six indeterminate quantities can be computed with sufficient accuracy. The uncertainty in finding the mass of the rings arises chiefly from the lack of good values of the masses of the satellites. These masses must be found from the mutual perturbations of the satellites. Substituting the values of the masses of the satellites determined by Professor H. Struve, the principal coefficient depending on the figure of the planet was assumed to be 0.0222. The mass of the rings is $1/7092$. It is probable that Struve's masses of the satellites are too small, and the above mass of the rings would be too great.

Saturn will soon return to our northern skies, and it is hoped that further observations and their discussion will give good values of the constants of this interesting system.

Professor John A. Eiesland, Thiel College, presented two papers:

I. "On a class of real functions to which Taylor's theorem does not apply."

As an example of a function to which Taylor's expansion does not apply Cauchy gave $f(x) = e^{-1/x^2}$. Pringsheim has the following series

$$f(x) = \sum \frac{(-1)^\nu}{\nu!} \frac{1}{a^\nu x + 1}, \quad a > 1 \text{ and real,}$$

which expanded gives rise to the series

$$\phi(x) = e^{-1} - e^{-a}x + e^{-a^2}x^2 - \dots \pm e^{-a^n}x^n \mp \dots$$

This series is convergent throughout the finite portion of the plane, while $f(x)$ in the neighborhood of $x = 0$ has an infinite number of poles $x = 0$. There is a class of functions of this kind which are distinguished by the additional property that they give rise to identically the same expansion. Such a function is

$$f(x) = \frac{\sum (-1)^\nu \left(\frac{\pi}{2}\right)^{2\nu+1}}{(2\nu+1)!} \cdot \frac{1}{1 + a^{2\nu+1}x},$$

which expanded becomes

$$\phi(x) = \sin \frac{\pi}{2} - \sin \frac{a\pi}{2}x + \sin \frac{a^2\pi}{2}x^2 - \dots \quad (a > 1).$$

If a is any even integer,

$$\phi_1(x) = 1,$$

if a is an odd integer of the form $4n + 1$,

$$\phi_2(x) = 1 - x + x^2 - x^3 + \dots = \frac{1}{x+1},$$

and if a is an odd integer of the form $4n - 1$,

$$\phi_3(x) = 1 + x + x^2 + \dots = \frac{1}{1-x}.$$

Hence all functions $f(x)$ having *different* poles may give rise to the same expansion in Taylor's series. The reason for this singular phenomenon must be sought in the periodicity of the coefficients in the expanded series. Another class of functions all having the *same poles* but differing in the values assigned to the coefficients present the same feature. Functions of the type

$$f(x) = \sum \frac{c_v}{1 + a_v x} \quad (\lim a_v = \infty)$$

may be expanded in a power series $A_0 + A_1x + A_2x^2 + \dots$ in which

$$A_0 = \sum_0 c_v, \quad A_1 = \sum_0 c_v a_v, \dots, \quad A_n = \sum_0 c_v a_v^n.$$

Now this system of equations may be solved for the unknown c_v by a method employed by Poincaré; * the system of values obtained is not unique, in fact as Poincaré shows

$$c_0 a^0, c_1 a, c_2 a^2, \dots, c_v a^v, \dots, -1$$

also form a system of solutions, and more generally

$$c_0 a^{0P}, c_1 a^P, c_2 a^{2P}, \dots, c_v a^{vP}, \dots, (-1)^P$$

are solutions of the system. But to all the different functions $f(x)$ that may thus be constructed there corresponds only one

* Bull. de la Soc. Math. de France, vol. 8.

expansion $\phi(x)$ which of course does not represent any of the functions.

II. "On a class of transcendental functions with line singularities."

Borel has given the function

$$\phi(z) = \sum b_\nu x^{c_\nu} \quad \left(\frac{c_\nu - c_{\nu-1}}{c_\nu} > N \right)$$

as an example of a class of functions which with all their derivatives remain finite and continuous within the unit circle as well as on it. To this class of transcendentals the following type may be added:

$$F(z) = \prod \frac{z - (1 + a_\nu)e^{2\pi i \nu a}}{z - (1 + b_\nu)e^{2\pi i \nu a}},$$

where $\lim a_\nu = 0$, $\lim b_\nu = 0$, and $a =$ an incommensurable number. The circle of radius unity is an essential singular line for all such functions, and this with respect to zeros as well as poles which are situated outside the circle, this curve being the point limit of poles and zeros. A more general type is

$$\Phi(z) = \prod \frac{z - (1 + a_\nu)e^{\nu i a}}{z - (1 + b_\nu)e^{\nu i a}} e^{\psi_\nu(z)},$$

where

$$\begin{aligned} \psi_\nu(z) = & \frac{(a_\nu - b_\nu)e^{\nu i a}}{z - (1 + b_\nu)e^{\nu i a}} + \frac{1}{2} \left[\frac{(a_\nu - b_\nu)e^{\nu i a}}{z - (1 + b_\nu)e^{\nu i a}} \right]^2 + \dots \\ & + \frac{1}{\nu - 1} \left[\frac{(a_\nu - b_\nu)e^{\nu i a}}{z - (1 + b_\nu)e^{\nu i a}} \right]^{\nu-1}. \end{aligned}$$

All these functions can be proved finite and continuous within as well as on the unit circle together with all their derivatives up to any order as high as we please. They admit of no analytical expansion beyond this circle; the zeros and poles approach the unit circle in a narrowing spiral, but none of these singularities are situated on the curve.

"On a general method of subdividing the surface of a sphere into congruent parts:" Mr. Harold C. Goddard, Amherst College.

The problem was incidental to the practical problem of constructing a steel sphere one hundred feet in diameter, in con-

nection with a new method of mounting a telescope, as outlined in an article in the *American Journal of Science* for June, 1902, by Professor David P. Todd.

If a regular dodecaedron be inscribed in a sphere, planes determined by the center and each edge of the dodecaedron cut out on the sphere twelve equal regular spherical pentagons. If the vertices of each pentagon be connected with its center by arcs of great circles the surface of the sphere is divided into sixty congruent isosceles spherical triangles, whose angles are determined as 60° , 60° and 72° .

Professor J. Burkitt Webb, Stevens Institute of Technology, presented two papers:

I. "A possible new law in the theory of elasticity."

II. "Displacement polygons."

The first of these papers was presented by abstract, and the second by title. The law referred to in the first is: "If the forcible change of the distance between two points in an elastic system changes the distance of two other points by a certain amount, then the same force applied to alter the distance of the two other points will change the distance of the first two points by the same amount."

"On extracting roots of numbers by subtraction," Dr. Artemas Martin, Washington, D. C.

This paper is a development and extension of an article, published in the *Philosophical Magazine* for September, 1880, by the Rev. F. H. Hummell. Dr. Martin shows how to determine any root of any perfect power by subtracting from the given power a certain series of subtrahends. If the given number is not a perfect power that fact is revealed at a close of the operation. The last remainder is always equal to the root sought, if the given number is a perfect power, and the number of subtractions is less by unity than the root sought. The general formula for the n th subtrahend of the m th root is

$$S_n = (n + 1)^m - (n^m + 1),$$

or
$$S_n = S_{n-1} + (n + 1)^m + (n - 1)^m - 2n^m.$$

The paper will be published in the *Mathematical Magazine*.

"On the positions of the northern circumpolar stars:" Professor Milton Updegraff, Washington, D. C.

The questions discussed by Professor Updegraff were :

(1) The importance of accurate knowledge of the positions of the fixed stars, and the greater attention, comparatively speaking, which has hitherto been paid to the places of the equatorial stars.

(2) The fundamental work on the circumpolar stars of the Greenwich and Pulkowa observatories. Differential work done by Groombridge, Carrington and Schwerd. Later work done at Greenwich and Pulkowa, and in America. Observations of circumpolar stars requested by Auwers.

(3) By observing the circumpolar stars at both upper and lower culminations on the same day, their places can be determined so that the results are practically fundamental. The fainter stars can be observed at both culminations on the same day, during the fall and winter months in middle latitudes, by observing a certain list of stars in the early morning and again early in the evening. Instrumental errors are largely eliminated, as also are errors of personal equation and of the refraction tables. The periodic variation of the position of the pole may be determined and also eliminated from the observed places. The formula for reducing the observations is the following well known modification of Bessel's formula for the reduction of observations in right ascension,

$$\alpha = T + \Delta T + m + (n + c) \tan \delta + c (\sec \delta - \tan \delta).$$

(4) The desirability was pointed out of an extended series of fundamental observations of both equatorial and circumpolar stars for increasing, for example, our knowledge of the motions and distribution of the stars in space. Such a series of observations would be useful in constructing a general catalogue of fundamental stars, numbering several thousand, and selected with reference to their suitability to serve as standard points of reference.

(5) In conclusion certain advantages in the location of the Naval Observatory in Washington for fundamental work on the fixed stars were stated. Washington is twelve and one half degrees in latitude south of Greenwich, twenty degrees south of Pulkowa, and seven and one half degrees south of the new branch of the Pulkowa Observatory at Odessa. It is however far enough north so that the northern circumpolar stars can be advantageously and accurately observed. For these reasons it may be said that perhaps no better location, as

regards latitude, could be found for supplementing the fundamental work done at Greenwich, Pulkowa and the Cape of Good Hope. The climate at Washington is very good for work of this kind. The local conditions are also favorable and are likely to remain so.

"The definite determination of the causes of variation in level and azimuth of large meridian instruments," Professor G. W. Hough, Dearborn Observatory, Evanston, Ill.

Professor Hough went into an elaborate discussion of several series of observations upon the causes and effects of this variation with different styles of mounting. His conclusion was that stone piers give the best results. The paper gave rise to some spirited discussion.

"Some theorems on ordinary continued fractions," Professor Thomas E. McKinney, Marietta College.

Let D be any positive integer not a perfect square and let its square root be represented by an ordinary continued fraction. Professor McKinney determines the form of D so that the continued fraction representing its square root may have a period with one, two, three or four elements and applies the results to the determination of the number of reduced forms in the class to which the indefinite quadratic form $(1, 0, D)$ belongs.

"On the forms of sextic scrolls of genus one," Dr. Virgil Snyder, Cornell University.

Dr. Snyder applied the same methods to find the forms of sextic scrolls of genus 1 as he previously employed in the study of the unicursal surfaces. Thirty-three types are found, ten of which have a multiple conic. The same method is being applied to sextic scrolls of genus greater than 1. The complete paper will be published in the *American Journal of Mathematics*.

"Transformation of the hypergeometric series," Professor Edgar Frisby, U. S. Naval Observatory.

If in the differential equation of the second order connecting the elements of the hypergeometric series

$$P = 1 + \frac{a\beta}{\gamma}x + \frac{a\beta(a+1)(\beta+1)}{1 \cdot 2 \cdot \gamma \cdot (\gamma+1)}x^2 + \text{etc.}$$

$x^a P'$ be substituted for P , new relations are obtained in which

P' takes the place of P , and the new elements are functions of the original elements. μ is determined from the condition that the new series must be of the same general form as the old. If, in addition, x be replaced by $1/x$ another series is obtained. From these two new series, by proper substitution of the new derived elements, are obtained, almost by inspection, the twenty-four different series ordinarily given in works on differential equations.

EDWIN S. CRAWLEY,
Secretary.

UNIVERSITY OF PENNSYLVANIA.

SECOND REPORT ON RECENT PROGRESS IN THE THEORY OF GROUPS OF FINITE ORDER.

BY PROFESSOR G. A. MILLER.

(Read before Section A of the American Association for the Advancement of Science, Pittsburg, July 2, 1902.)

THE main extensive treatments of this theory which have appeared during the four years since my first report was presented before this Section are: The articles in volume I of the *Encyclopädie der mathematischen Wissenschaften* on "Endliche discrete Gruppen," "Galois'sche Theorie mit Anwendung," and "Endliche Gruppen linearer Substitutionen," by Burkhardt, Hölder, and Wiman respectively; Weber, *Lehrbuch der Algebra*, second edition, volume 2, 1899; Bianchi, *Lezioni sulla teoria dei gruppi di sostituzioni*, 1899; * Echegaray, *Lecciones sobre resolucion de ecuaciones y teoria de ecuaciones*, 1899; † Netto, *Vorlesungen über Algebra*, volume 2, 1900; Pierpont, "Galois theory of algebraic equations," ‡ 1900; Dickson, *Linear groups*, 1901; Burnside and Panton, *Theory of equations*, volume 2, 1901.

In the present, as in my first report, it is my intention to avoid, as far as practicable, the consideration of those recent advances which have received considerable attention in these

* Printed edition of the work lithographed in 1897.

† Reviewed in *L'Enseignement Mathématique*, vol. 2 (1900), p. 227.

‡ *Annals of Math.*, 2d series, vols. 1 and 2.

articles and treatises, and to devote most of the space to a few recent developments which seem to offer inviting fields of investigation. It is hoped that this report will soon be supplemented in some directions, like the first report the usefulness of which was greatly extended by Professor Dickson.*

For several years Dr. Steinitz, of Charlottenberg, has had under consideration a report on the theory of groups of finite order for the Deutsche Mathematiker-Vereinigung. In a recent letter he expressed the hope of having this report ready for the next meeting of the Vereinigung, which will be held at Karlsbad during the coming September in connection with Section I of the Gesellschaft Deutscher Naturforscher und Aerzte. Dr. Easton, of the University of Pennsylvania, has prepared a bibliography of the theory of substitutions together with a collection of theorems and definitions, which is now in press.

§1. ABSTRACT GROUPS.

If the three numbers l, m, n are the orders of three operators one of which is a product of the other two, it is well known that the group generated by any two of these operators is completely determined by l, m, n provided at least two of these numbers are 2; or one is 2, another 3, while the third is 3, 4, or 5. The groups which may be defined in this way, together with those which are cyclic, are known as the groups of genus zero, and have received a great deal of attention as they are the groups of the rotations into themselves of the regular solids and hence enter prominently into many geometric considerations.

It has recently been proved † that three operators L, M, N whose orders are fixed numbers l, m, n respectively can always be so selected as to generate any one of an infinite system of groups of finite order whenever each of these fixed numbers exceeds unity and the three do not satisfy one of the special conditions mentioned in the preceding paragraph. The proof of this theorem is based upon the fact that it is always possible to find three operators such that one of them is a product of the other two and that their orders are the three arbitrary numbers l, m, n . Both of these theorems were proved by means of sub-

* Dickson, "Report on the recent progress in the theory of linear groups," BULLETIN, vol. 6 (1899), p. 13.

† *Amer. Jour. of Math.*, vol. 24 (1902), p. 96; cf. Dyck, *Math. Annalen*, vol. 20 (1882), p. 34.

stitutions. It would appear desirable that more abstract proofs should be given, especially since the published proofs are based upon considerations which are not commonly employed.

The infinite systems of groups which belong to a particular set of values of l, m, n have received very little attention except in the three cases where these numbers are such as to give rise to groups of genus one. These cases have been studied geometrically by Dyck * and Burnside.† Recently they have been studied from the purely analytic standpoint and some interesting common properties have been found.‡ Perhaps the most important of these is the fact that all of them contain an abelian commutator subgroup.

The groups of genus one which are not included in the three cases mentioned in the preceding paragraph are those which may be generated by three operators of order two whose continued product is also of order two, but which cannot be generated by two operators of order two. All of these groups may be constructed by adding to *any* abelian group A , having just two independent generators, an operator of order two which transforms each operator of A into its inverse. In this way a group is obtained whose order is twice the order of A and which contains only operators of order two besides those of A . This is the simplest category of groups of genus one and has many properties in common with the dihedral rotation groups. It may be observed that none of these groups can be generated by less than three operators. This category includes only one abelian group, viz., the group of order 8 and of type (1, 1, 1).

From what precedes it is clear that little is known about the groups generated by the two operators L and M when the three numbers l, m, n are given. A closely related problem which has received but little attention is the study of the groups in which the order of every operator is a divisor of a fixed number k . The simplest case, viz., when $k = 2$, has been completely solved.§ There is just one such group of order 2^α for every value of α . This is the only value of k for which all the possible

* *Loc. cit.*

† *Theory of groups of finite order*, 1897, p. 293.

‡ *Quar. Jour. of Math.*, vol. 33 (1901), p. 76; *Annals of Math.*, vol. 3 (1901), p. 40; The case when the groups are generated by two operators of order 4 whose product is of order 2 was considered by Manning in a paper read at the first meeting of the San Francisco section of the Amer. Math. Soc., May, 1902.

§ *Quar. Jour. of Math.*, vol. 28 (1896), p. 208.

groups are abelian. When $k = 3$ it is very easy to prove that all the conjugate operators are commutative. Moreover, it is evident that this property is not common to all the groups for any value of k greater than 3.

Recently Burnside has made a study of all the possible groups when $k = 3$, and also of the special case when $k = 4$ and the groups have but two generators.* The more general case when k is any prime p seems to present considerable difficulty. The order of such a group is evidently a power of p . In the special case when there is an abelian subgroup whose order is the order of the entire group divided by p it is not difficult to determine all the possible groups.† Even for p^5 , $p > 3$, there are groups which contain only operators of order p without containing the abelian group of order p^4 .

The questions considered in the preceding two paragraphs may be regarded as special cases of the problem to determine all the groups which are conformal with systems of abelian groups. If two abelian groups are conformal, i. e., if they contain the same number of operators of each order, they cannot be distinct; but with respect to the non-abelian groups the matter is entirely different. Recently all the possible abelian groups which are conformal with non-abelian groups have been determined,‡ but the total number of non-abelian groups which are conformal with fixed types of abelian groups are known in only a few cases. For instance, it is known that there is only one non-abelian group which is conformal with the abelian group of order p^m and of type $(m - 1, 1)$. The only other types of order p^m for which all the conformal non-abelian groups are known are $(m - 2, 2)$ and $(m - 2, 1, 1)$.§

If p^a is the highest power of the prime number p which divides the order of a group then the group must contain $1 + kp$ conjugate subgroups of order p^a according to Sylow's theorem. Moreover, it is known that there are groups for all values of p when k is either 0 or 1. Burnside has recently examined the cases when k is either 2 or 4 and found that in both of these cases the possible values of p are greatly restricted. His results are, when $k = 2$, then $1 + 2p$ is either a prime or a power of 3; and when $k = 4$, then $1 + 4p$ is either a prime, 9,

* Burnside, *Quar. Jour. of Math.*, vol. 33 (1902), p. 230.

† BULLETIN, vol. 8 (1902), p. 39.

‡ BULLETIN, vol. 8 (1902), p. 154.

§ *Loc. cit.*, p. 206.

or a power of 5. The case when $k = 3$ seems to present greater difficulties but it is easy to see that there are also limitations on the form of p for this value of k . *

Most of these advances in the theory of abstract groups have been made by means of substitution groups. When this method is employed comparatively little use is made of definitions, as all the steps can be conveniently illustrated by means of concrete examples. For some purposes it is however desirable to proceed more abstractly. This seems to be especially the case in a systematic presentation of the subject. Hence it becomes desirable that the definitions should be so formulated as to be as useful as possible. It is also desirable to avoid redundancies as these mar the beauty of the presentation. Recently Dr. Huntington has pointed out redundancies in the definitions usually given and has formulated definitions which do not involve these objectionable features. † Professor Moore has also presented a note on this subject. ‡

Scarpis has recently considered some properties of commutators and of commutator subgroups. He proves by a method which differs somewhat from the one employed earlier, that the important question of solvability can readily be decided by means of commutator subgroups. § Although he gave a general reference to the earlier article in which this theorem is proved, yet one would naturally infer in reading his article that he considered the theorem as new. One might also question the propriety of naming these groups after Dedekind, since he published very little in regard to them, and, moreover, was not the first to make their properties known. ||

The interesting question whether every operator of a commutator subgroup is a commutator has been answered in the negative in volume 2, page 133, of the second edition of Weber's Algebra. Dr. Fite has found some conditions which are sufficient for the existence of such operators, and has published a group which satisfies these conditions. ¶ Wundt proved a very interesting theorem in regard to the special class of soluble groups which possess a principal series of composition such that

* Burnside, *Messenger of Mathematics*, vol. 31 (1901), p. 77.

† Huntington, *BULLETIN*, vol. 8 (1902), p. 296.

‡ *BULLETIN*, vol. 8 (1902), p. 373.

§ Scarpis, *Giornale de Matematiche*, vol. 39 (1901), p. 376; cf. *BULLETIN*, vol. 6 (1899), p. 105.

|| Cf. Frobenius, *Berliner Sitzungsberichte*, 1896, p. 1348; Cantor, *Geschichte der Mathematik*, vol. 2 (1900), p. 811.

¶ *BULLETIN*, vol. 6 (1900), p. 181.

all of the corresponding factor groups are of prime order. Whenever this condition is satisfied he proved that the commutator subgroup must be the direct product of groups whose orders are powers of primes.* This proof is partly based upon the fact that every group contains an invariant subgroup (which may be the identity) which is the direct product of groups whose orders are powers of primes and includes all other invariant subgroups that have this property.

The most extensive work on the enumeration of abstract groups which has appeared during the last few years seems to be the *Enumeration des groupes d'opérations d'ordre donné*, 1901, 128 p., lith., by Le Vavasseur. This work, which is published by Hermann, Paris, was noticed by the writer too lately to permit an examination before writing this report. The same author has recently published less extensive enumerations of the groups of order p^2q^2 and also those of order $16p$, p and q being primes and p being odd.† Western has considered the groups of order p^3q ,‡ but no one seems to have determined all the groups in the remaining case where there are four prime factors, viz., when the order is p^2qr . The enumeration of groups of low orders has been extended to those of order 64.§

Dickson has embodied a large number of the results of his extensive investigations in his *Linear Groups*. The last chapter of this work is devoted to a summary of known systems of simple groups. It is observed that two of these systems involve simple groups of the same order that are not isomorphic. The object of some of his more recent papers is to show that several branches of group theory may be correlated by means of transformations in a given domain of rationality. The special simple group of order 504 has been defined more abstractly by Burnside in the *Mathematische Annalen*, volume 52 (1899), page 174. Fricke considers this group from a different standpoint in the same volume, page 321.

In a memoir published in *Bihang till Kongl. Svenska Vetenskaps Akademiens Handlingar*, volume 25 (1900), Wiman determined all the subgroups of a doubly infinite system of simple groups. The article begins with a study of the Galois imaginaries and the groups of finite order defined by congruences.

* Wundt, *Math. Annalen*, vol. 55 (1901), p. 479.

† La Vavasseur, *Comptes rendus*, Paris, vol. 128 (1899), p. 1152; vol. 129, p. 26.

‡ Western, *Proc. of the London Math. Soc.*, vol. 30, p. 209.

§ *Quar. Jour. of Math.*, vol. 30 (1898), p. 243.

Bauer, in a series of articles published in *Nouvelles Annales*,* proved a number of theorems relating to the number of subgroups of particular orders that can occur in a group whose order is known. Most of these theorems are based upon those proved by Frobenius in the *Berliner Sitzungsberichte* of 1895.

While some questions in regard to abelian groups, for instance those which relate to the number and the types of groups of a fixed order, have been completely solved yet there is a large number of others which have received but little attention. Steinitz has recently considered one of the latter, viz., the determination of how many times a group γ is contained in the product $\alpha\beta$ of two abelian groups. In other words, to determine the number of subgroups α' such that α' is isomorphic with α and γ/α' is isomorphic with β . Representing this number by $t(\gamma; \alpha, \beta)$ Steinitz investigated some of the properties of this function and determined all these groups when β is cyclic or has just two independent generators.†

On the occasion of the jubilee of Sir G. G. Stokes, Burnside presented a brief memoir on the groups of even order which contain no operator of even order except those of order 2. He found that these groups may be divided into three distinct classes which are represented by the dihedral, tetrahedral, and the icosahedral rotation groups. In a more recent paper communicated to the London Mathematical Society he considers the groups in which every two conjugate operators are permutable and proves that the necessary and sufficient condition that the order of such a group is finite is that the generators are of finite orders. The case when the order is a power of 3 is exceptional.‡

§ 2. HOLOMORPHISMS.

A simple isomorphism of a group G with itself has been called a holomorphism or an automorphism of G .§ The totality of the holomorphisms of G correspond to a group I known as the group of isomorphisms. When G is transformed by its own operators one or more holomorphisms are obtained. The totality of the holomorphisms that can be obtained in this way correspond to an invariant subgroup I_1 of I , which is known as

* Vol. 19 (1900), pp. 59, 508, 509.

† Steinitz, *Jahresbericht der Deutschen Math.-Vereinigung*, vol. 9 (1901), p. 80.

‡ *Nature*, vol. 66 (1902), p. 71.

§ *Annals of Math.*, vol. 2 (1901), p. 78; Frobenius, *Berliner Sitzungsberichte*, 1901, p. 1324.

the group of cogredient isomorphisms. When I is abelian, G is said to be metabelian and vice versa. A characteristic property of metabelian groups is that all their commutators are invariant.

The metabelian groups constitute a special case of a category of groups which seems to have been first studied by Ahrens; viz., those in which one arrives at the identity by forming the successive groups of cogredient isomorphisms.* It is easy to prove that all the groups of this category are the direct products of groups whose orders are powers of single prime numbers and vice versa.† Dr. Fite has recently made an extensive study of the properties of metabelian groups and some of his results have been published in the form of abstracts. His memoir on these groups is to appear soon in the *Transactions of the American Mathematical Society*.

The term metabelian seems appropriate since these groups have many properties which differ but slightly from those of abelian groups. In the development of their theory it is convenient to make prominent use of the commutator subgroup and the group of cogredient isomorphisms. When the former is cyclic and of order p^a , p being any prime, the latter contains an even number of independent generators, and the corresponding metabelian groups are the direct products of abelian groups and a metabelian group of order p^m .‡ The Hamiltonian groups constitute a special case of those metabelian groups whose group of cogredient isomorphisms is the four-group. The theory of metabelian groups clearly involves the determination of all the abelian groups which can be groups of cogredient isomorphisms. Most of the results in this direction are found in the memoir mentioned in the preceding paragraph.

A number of important properties of a group are exhibited in its group of isomorphisms. Comparatively little progress has been made in the theory of the simple isomorphisms of a group with itself. One of the most important steps in this direction has been the determination of all the invariant operators of I when G is abelian. These operators correspond to the holomorphisms of G in which every subgroup corresponds to itself. If this condition is satisfied every operator of G corresponds to

* Ahrens, *Leipziger Berichte*, vol. 49 (1897), p. 616; cf. Fite, *BULLETIN*, vol. 8 (1902), p. 236.

† Burnside, *Theory of groups*, 1897, p. 115; cf. Loewy, *Math. Annalen*, vol. 55 (1901), p. 67.

‡ *BULLETIN*, vol. 7 (1901), p. 86.

the same power of itself.* Hence the number of invariant operators of I is the totient of the maximum order of any operator of G .

These results have been very materially extended by Mr. Young in his recent article "On the holomorphisms of a group."† He employs for the first time the convenient term *α -holomorphism* to denote a simple isomorphism of a group with itself such that every operator corresponds to its α th power, and inquires into the necessary and sufficient condition that a non-abelian group may admit α -holomorphisms besides the identical one. He finds that a group cannot admit an α -holomorphism unless the $(\alpha - 1)$ th power of every operator is invariant. When the order of the group is a power of a prime these α -holomorphisms correspond to invariant operators in the group of isomorphisms but it is not always possible to obtain all the invariant operators of the latter group in this manner.

In any holomorphism each operator corresponds to itself multiplied on the right by some operator of the group. The totality of these multiplying operators do not necessarily form a group. For instance, if the symmetric group of order 6 is transformed by one of its operators of order three, the holomorphism will be such that the multiplying operators do not form a group; but these operators form a group when the holomorphism may be obtained by transforming by an operator of order two. In every holomorphism of an abelian group A these multipliers always form a group‡ and it is frequently convenient to consider the holomorphisms of A from this standpoint.

Not only do the multipliers of the preceding paragraph form a subgroup of A but this subgroup (which may coincide with A) is also isomorphic with A in every possible simple isomorphism of A with itself. If the order of this subgroup is either p or $2p$; the resulting holomorphism of A will correspond to an operator of order p , $2p$, or $\frac{p-1}{\alpha}$ (α being a divisor of $p-1$) in the group of isomorphisms I of A . If, moreover, each operator of this subgroup corresponds to itself in such a holomorphism, the order of the corresponding operator of I is clearly equal to the order of the largest operator of the subgroup. Very little has been done towards the development of the theory of holomor-

* *Transactions Amer. Math. Soc.*, vol. 2 (1901), p. 260.

† Young, *Ibid.*, vol. 3 (1902), p. 186.

‡ BULLETIN, vol. 6 (1900), p. 337.

phisms from this standpoint. It may be possible to find simple conditions which are necessary and sufficient to establish holomorphisms of non-abelian groups in this manner. Such conditions would be of considerable interest.

One of the simplest ways of obtaining a holomorphism of a non-abelian group G is to make each of its operators A correspond to its transform with respect to any one of its operators B . In calling attention to a slight error in Burkhardt's article in the *Encyklopaedie*, Loewy writes this transform BAB^{-1} instead of $B^{-1}AB$.^{*} Moreover Loewy writes $BA^{-1}B^{-1}$ instead of $BA^{-1}B$ as given by Burkhardt. The latter is, however, a very insignificant error; while the former cannot be regarded as an error since some good writers have used BAB^{-1} for the transform of A with respect to B ;† but the majority of writers (including Jordan, Lie, Netto, Weber, Burnside, Bianchi) use the form $B^{-1}AB$. The form used by the minority seems to have decided advantages. For instance, in considering the expression

$$\begin{aligned}(AB)^n &= ABABABAB \dots \\ &= ABAB^{-1}B^2AB^{-2}B^3AB^{-3} \dots\end{aligned}$$

it is convenient to call BAB^{-1} the transform of A , especially if some of the commutators are not invariant. Although this is not a very important matter yet it would be of some interest to know which of these forms is the most advantageous, especially since transforms are so frequently employed.

If any group G is represented as a regular substitution group there is a totality of substitutions in the same elements as those involved in G which transform G into itself. These form a non-regular group known as the holomorph of G . Any one of the largest subgroups which do not involve one of the elements of this holomorph is simply isomorphic with the group of isomorphisms of G and every holomorphism of G can therefore be obtained by transforming G by some operator in such a subgroup. The substance of these remarks is found in my former report, page 245. It is repeated here since Burkhardt's statement on bottom of page 221 of the *Encyklopaedie* seems to imply that the holomorph of G is also its group of isomorphisms, which is evidently incorrect.

^{*} Loewy, *Math. Annalen*, vol. 55 (1901), p. 68.

† Dyck, *Math. Annalen*, vol. 22 (1883), p. 75. The present usage seems to be due to the influence of Jordan; cf. Hagen, *Synopsis der höheren Mathematik*, vol. 1 (1891), p. 285. Cauchy and Serret used the form BAB^{-1} , which was called the derivative of A by Betti.

It is known that cyclic groups are the only abelian groups whose groups of isomorphisms are abelian.* No one seems to have investigated the question whether a non-abelian group can have an abelian group of isomorphisms. If this question should be answered in the negative it will follow that the system of groups which Weber calls the most important example of abelian groups of finite order † is composed of all the possible commutative groups of isomorphisms. Since the group of cogredient isomorphisms of a non-abelian group cannot be cyclic, it follows that a cyclic group cannot be a group of isomorphisms unless its order is of the form $p^a(p-1)$, p being an odd prime.

Groups which have the same group of isomorphisms may differ very much from each other. The totality of such groups has been determined in only a few cases while no one seems to have determined all those which have the same group of cogredient isomorphisms. Some of the simplest cases would evidently present very few difficulties. For instance, when the group of cogredient isomorphisms of G is of order p^2 it must be of type $(1, 1)$ and G must be the direct product of abelian groups and a non-abelian group P of order p^m whose group of cogredient isomorphisms I_1 is of order p^2 . Moreover, P contains an abelian subgroup of order p^{m-1} which includes the p^{m-2} invariant operators. When this abelian subgroup is cyclic it is well known that there is only one such group. When it is of type $(1, 1, 1, \dots)$ there are clearly three such groups if $p > 2$ and $m > 3$; one is conformal with the abelian group of type $(1, 1, 1, \dots)$ while the others are conformal with the one of type $(2, 1, 1, \dots)$.

§ 3. SUBSTITUTION GROUPS.

Maillet and Burnside have considered the transitive groups of degree n and of class $n-1$. In his thesis entitled, "*Recherches sur les substitutions, et en particulier sur les groupes transitifs*," Paris, 1892, Maillet divides these groups into two categories. The first of these included all those which contain a regular group of order n , while the second included those which do not have this property. He states that in all the particular cases which he has examined there is no primitive group belonging to the second category. In several later articles pub-

* *Transactions*, vol. 1 (1900), p. 397.

† Weber, *Lehrbuch der Algebra*, vol. 2 (1899), p. 60.

lished in the *Bulletin* of the French Mathematical Society * he extends a number of his results in regard to these groups.

Burnside takes up the same question both in his *Theory of Groups* and also in an article published in the *Proceedings of the London Mathematical Society*.† In the latter he proves that every transitive group of degree n and of class $n - 1$ contains a regular group of order n whenever $n \equiv 81,000,000$. Recently Frobenius proved ‡ that this theorem is true for all values of n and thus established one of the most useful recent theorems in the theory of substitutions. Since the order of each operator in the regular group is a divisor of n while the order of every other operator divides $n - 1$ it follows that there can be only one subgroup of order n .

During the last four years the enumerations of intransitive, imprimitive, and primitive groups have each been extended to one larger degree. The list of the intransitive groups of degree eleven includes 1,492 distinct substitution groups, which is about 500 more than the number for degree 10.§ Nothing seems to have been done towards giving general formulas by means of which the number of these groups can be determined without trial except for the cases when the order is either the product of two distinct primes or when it is 4. Miss Martin published a list of the imprimitive groups of degree 15 and also one of the primitive groups of degree 18.|| She remarks that Dr. Kuhn has also made a study of the imprimitive groups of degree 15 and has arrived at results which differ greatly from those which she obtained.

The efforts to discover a simple group of odd composite order have led to the study of primitive groups of such an order, for every simple group is simple isomorphic with one or more primitive substitution groups. Burnside proved the interesting theorem that the maximal subgroup which does not involve one of the elements of such a group must contain an even number of transitive constituents.¶ Dr. Rietz has also established a number of theorems relating to this maximal subgroup** which seems to present one of the most accessible approaches towards

* Vol. 25 (1897), p. 16 ; vol. 26 (1898), p. 249.

† Vol. 32 (1901), p. 240.

‡ Frobenius, *Berliner Sitzungsberichte*, 1901, p. 1220.

§ Miller and Ling, *Quar. Jour. of Math.*, vol. 32 (1901), p. 342.

|| Miss Martin, *Amer. Jour. of Math.*, vol. 23 (1901), p. 259.

¶ Burnside, *Proc. of the London Math. Soc.*, vol. 33 (1901), p. 165.

** BULLETIN, vol. 8, pp. 17 and 275.

the theory of primitive groups. By means of these theorems and the important theorem, due to Burnside, that there is no primitive group of odd composite order and of a prime degree p that contains more than one subgroup of order p , Dr. Rietz has determined all the primitive groups of odd order whose degree does not exceed 242. All these groups are solvable.

It is well known that the average number of elements in all the substitutions of a transitive group of degree n is $n - 1$. Hence every transitive group must contain at least $n - 1$ substitutions of degree n to make up the average for the identity. In general, for each of its substitutions of degree $n - \alpha$ there must be $\alpha - 1$ substitutions of degree n . Dr. Rietz has recently proved * a closely related theorem to the effect that a primitive group of degree n and of order g contains more than $g/x + 1$ substitutions of degree less than n , x being the number of transitive constituents in a maximal subgroup of degree $n - 1$. This theorem is closely related to the investigations of Jordan, Bochert, Maillet, and others in reference to the class of a primitive group.

Although the primitive groups of low classes present much greater difficulties than those of low degrees, yet it is easy to determine all the primitive groups of a prime class p while no one has yet succeeded to determine those of any general degree.† In a memoir published in *Videnskabs-Selskabets Skrifter*, 1897, Sylow considers the groups of degree p and of order $p(p + 1)\pi$, π being a divisor of $p - 1$. He proved that there are no groups of degree p and of order $p/2(p^2 - 1)$ except in the cases when p has one of the values 5, 7, 11. In each of these special cases there is only one such group.

Lombardi has recently developed a number of theorems in regard to imprimitive groups.‡ Although his results differ but slightly from well-known theorems yet no references are given. Dr. Kuhn has made a much more extensive study of the theory of these groups. Only abstracts of his memoir have appeared.§ In the first of these it is pointed out that there are groups in addition to those noted by Dyck in *Mathematische Annalen*, volume 22, which cannot be represented in any non-regular imprimitive form. In connection with Wilkinson's article in regard to Jordan's theorem on the constancy of the factors of

* BULLETIN, vol. 8, p. 276.

† Jordan, *Liouville*, vol. 17 (1872), p. 363.

‡ Lombardi, *Giornale di Matematiche*, vol. 39 (1901), p. 134.

§ BULLETIN, vol. 7 (1901), pp. 5 and 116.

imprimitivity * it should be observed that Jordan himself noted the error in his *Traité des Substitutions*. He published a note in regard to the matter in *Giornale di Matematiche*, volume 10 (1872), page 116.

An abstract group is generally said to be represented by a substitution group if the two groups are simply isomorphic. The term is used in this sense when it is said that every abstract group can be represented as a regular substitution group. Quite recently Burnside published an article in which he uses this term in a more general sense in accord with the phraseology used by Frobenius in the study of linear substitutions. Any substitution group to which an abstract group has an $(\alpha, 1)$ isomorphism is said to be a representation of the abstract group.†

According to this use of the term 1, (ab) is a representation of every group which involves a subgroup of half its order. It appears to the writer somewhat unfortunate to use a term which had such definite meaning in the theory of substitutions, in this general way. It may be added that some of the statements of this article are not in accord with known results unless other terms are also given an unusual meaning. Lines 8–11 on page 161 may serve as an instance. Additional references to other articles bearing on the same question would have increased the usefulness of the article very materially.

Beginners in the study of the theory of substitutions are naturally much hampered by the different uses of the same term as well as by the number of different terms employed to express the same concept. Within the last few years Netto has perhaps made the most serious contribution towards increasing the latter difficulties by using such new terms as *conjug*, *conjugé complexe*, and *autojugen Theiler*.‡ The last of these terms is used for invariant subgroup so that the Germans can now express this concept by any one of the following six terms: *invariante*, *ausgezeichnete*, or *monotypische Untergruppe*; *autojuger*, *eigentlicher*, or *Normaltheiler*. The first four and the last of these seem to be due to König, Lie, Frobenius, Netto, and Weber respectively, while *eigentlicher Theiler* seems to have been derived from *décomposition propre*, used by Galois § who introduced the concept of invariant subgroup.

* Wilkinson, *Quar. Jour. of Math.*, vol. 30 (1898), p. 157.

† Burnside, *Proc. of the London Math. Soc.*, vol. 34 (1902), p. 159.

‡ Netto, *Vorlesungen über Algebra*, 1900, p. 328.

§ Liouville, vol. 11 (1846), p. 408.

Frobenius has employed the terms *inner* and *outer* automorphisms* instead of Hölder's terms *cogredient* and *contragredient* isomorphisms. These new terms have the advantage of being shorter and perhaps more natural than the older ones. Moreover, it seems desirable to have a term which explicitly states that a simple isomorphism is under consideration. This may be done by either of the new terms *holomorphism* or *automorphism*, which were introduced about the same time. Possibly it would be well to restrict the latter, as Frobenius has done, to express a simple isomorphism of a group with itself, while the former might express a general simple isomorphism.

It may be observed that the nomenclature here suggested is not free from objections. It is sometimes desirable to speak of a general isomorphism of a group with itself, and it would not appear unnatural to use the term *automorphism* for this purpose instead of restricting its use to a simple isomorphism of a group with itself. Moreover, Ritter, at the suggestion of Klein, has employed *holomorphism* for a general isomorphism and has used the term *isomorphism* to represent only a simple isomorphism. Burkhardt has followed this terminology in his *Encyklopädie* article.† It is open to the objection that the term *isomorphism* has been extensively used in a broader sense.

As the elements of a substitution group are not, in general, numbers, their laws of combination cannot be directly represented by relations between numbers. In a series of articles published in the *Mathematische Annalen* and in *Crelle*, Hoyer has developed a new theory for the algebraic solution of the problems of substitution groups.‡ These developments not only concern themselves with abstract group properties, such as the smallest number of generational relations and the finiteness of the order, but also with those relating to the representation as substitution groups, such as the degree of transitivity, the class, and the systems of imprimitivity. The general question of the determination of all the possible substitution groups of degree n is reduced to an algebraic one. Most of the developments are quite general and the results do not appear directly useful in advancing the theory along the older lines.

* Frobenius, *Berliner Sitzungsberichte*, 1901, p. 1324.

† Burkhardt, *Encyklopädie der Mathematischen Wissenschaften*, vol. 1, p. 220.

‡ Hoyer, *Math. Annalen*, vol. 49 (1897), p. 39; vol. 50 (1898), p. 499; vol. 51 (1899), p. 445; vol. 52 (1899), p. 550; *Crelle*, vol. 124 (1901), p. 102.

§ 4. GROUP CHARACTERISTICS.

The independent generators of any abelian group A may be represented by cyclic substitutions in distinct sets of letters. If the orders of these cyclic substitutions are $\rho_1, \rho_2, \dots, \rho_m$ the order of A is $\rho_1 \rho_2 \dots \rho_m$. The group generated by any one of these substitutions of order ρ_a is simply isomorphic with the ρ_a roots of unity. If the roots which may be associated with the different generating substitutions are regarded as independent numbers just as the substitutions are independent, these roots will generate a group which is simply isomorphic with A . The numbers which constitute the operators of this group are the *group characteristics* of A .

A more abstract definition of the characteristics of A is as follows: * With every operator s of A there is associated a number which may be regarded as a function of s and is denoted by $x(s)$. This number must satisfy the following conditions: (1) It does not vanish for any value of s , (2) the equation $x(ss') = x(s)x(s')$ is satisfied for every pair of operators in A . Two such numbers as x, x' are to be regarded as different when there is some operator s_1 in A such that $x(s_1) \neq x'(s_1)$. The numbers thus defined are the group characteristics of A and it is not difficult to deduce the properties mentioned in the preceding paragraph from these more abstract definitions. The characteristics of an abelian group were already employed by Lagrange and Dirichlet † and their abstract definition is due to Dedekind.

Since 1896 Frobenius has published a series of memoirs in which he has developed a theory of group characteristics for non-abelian groups, which reduces to the preceding when the group is abelian. ‡ He proves that the number of distinct characteristics is equal to the number of complete sets of conjugate operators of a group and that a characteristic of a given operator is the same as that of each of its conjugates. Moreover, each of the characteristics is the sum of roots of unity and the product of two characteristics can be represented as a linear function of all the characteristics of the group.

Weber gives a brief exposition of the theory of characteristics in the second edition of his *Algebra*, volume 2, 1899. More

* Cf. Weber, *Lehrbuch der Algebra*, vol. 2 (1899), p. 49.

† Frobenius, *Berliner Sitzungsberichte*, 1896, p. 985.

‡ *Loc. cit.*, 1896, pp. 985, 1343; 1897, p. 994; 1898, p. 501; 1899, pp. 330 and 482; 1900, pp. 303 and 516.

recently, Burnside has developed the same subject from a somewhat different standpoint in an article published in the *Proceedings of the London Mathematical Society*.* In two subsequent articles published in the same volume he has applied the theory in the study of some properties of groups of odd order. In the first of these papers he proves that a linear substitution of odd order is reducible whenever all the coefficients are real and that there is no simple group of an odd composite order and of a prime degree. The main result arrived at in the second of these papers is that if an odd prime p is the smallest factor of the order of a group there must be an invariant subgroup of index p unless either p^4 or p^3q , where q is a prime factor of $p^2 + p + 1$, is a factor of the order.

To facilitate the application of characteristics Frobenius developed two methods which are more convenient in many special cases than the general method developed in 1896. The first of these is based upon the relations which exist between the characteristics of a group and those of a subgroup.† The case when this subgroup is invariant leads to especially interesting results. Moreover, a characteristic can be directly obtained by this method in case the group is represented as a doubly transitive substitution group. The second method is based upon the composition of the characteristics and is illustrated by means of the alternating and the symmetric groups of degree 6 as well as by the tetrahedral, octahedral and icosahedral rotation groups.‡

The *Berliner Sitzungsberichte* for 1901 contains three papers by Frobenius on solvable groups. In the first two of these group characteristics are extensively used. The main theorem of the first paper is as follows: If a group of order fg , f and g being relatively prime, contains a subgroup of order f composed of invariant operators then it must be the direct product of this subgroup and a subgroup of order g . This theorem was afterwards proved without the use of group characteristics by de Seguer in *Comptes Rendus*, volume 134 (1902). The second of these papers contains, among many others, the interesting theorem mentioned in the preceding paragraph that a group of class $n - 1$ always contains a regular group of order n .

The principal theorem which is proved in the last of these three articles may be stated as follows: If p^λ is the highest power

* Vol. 33 (1901), p. 146.

† Frobenius, *Berliner Sitzungsberichte*, 1898, p. 501.

‡ *Loc. cit.*, 1899, p. 330.

of p which divides the order of a group H , and if no operator of H whose order is prime to p transforms a subgroup of order p^a into itself without being commutative with each one of its operators, then must H contain a subgroup of index p^a which is composed of all the operators of H whose orders are not divisible by p . Frobenius observes that it is possible to state this theorem somewhat more generally; but in this case the statement becomes still more complex and we shall not present it here.

STANFORD UNIVERSITY,
June, 1902.

SHORTER NOTICES.

Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance. By M. CURTZE. Erster Theil. *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, XII. Heft. Leipzig, Teubner, 1902. 336 pp. 16 Marks.

IT is a compliment to the monumental work of Professor M. Cantor that the activity in the field of the history of mathematics for the past twenty years has been almost entirely directed by him. The sole effort has been to supplement his work, to enter some of the innumerable doors which he has opened, to decipher the inscriptions upon the monument which he has erected. Hardly an article appears in the *Bibliotheca Mathematica*, relating to the period preceding the middle of the eighteenth century, that does not refer in some way to Cantor's work, and the *Abhandlungen* have been more or less under his direction for a quarter of a century.

Herr Curtze's latest contribution is an evidence in point, not merely in being dedicated to Professor Cantor on the occasion of his doctor's jubilee, but in that it elaborates certain details of his History for which elaboration scholars have been waiting.

Half of the work is given to the *Liber embadorum* of Abraham Savasorda (Sahib al Schorta, chief of the guards) as translated from the Hebrew by Plato of Tivoli in 1116, a treatise merely mentioned by Cantor.* The work has already been noticed by Curtze† as being one of the chief sources of

* *Vorlesungen*, vol. 2, p. 853.

† *Bibliotheca Mathematica*, vol. 1, 3d series (1900), p. 501.

Leonardo of Pisa's *Practica Geometriæ*, and Braunmühl has testified * to its importance in the history of trigonometry. Of course it has also been more or less known to other historians, particularly to Steinschneider † who has devoted so much attention to Jewish mathematics. ‡ The work itself, however, has been closed to the world at large until the appearance of the present edition.

It speaks well for the brotherhood of nations, as also of scholars, that the French government should permit the two manuscripts, which form the basis for the present edition, to be sent from the National Library and be placed at the disposal of Herr Curtze. As a result we have, on opposite pages, the Latin and German texts, the former with the variations in reading of the two manuscripts.

The most interesting fact relating to the treatise is that it shows at the same time the chief source of Fibonacci's work on geometry, and the originality of the latter. Both of these facts appear throughout the treatise, and are emphasized in the footnotes. While the work is devoted chiefly to geometry, it is also valuable as giving insight into the Spanish arithmetic of the period. It shows that, among the learned class, the subject was of the Boethian stamp, the Hindu numerals being unknown, and the definitions being mainly the conventional ones of the middle ages. *Mutabemini numeri* and *almugesem numeri* are new names, however, and the explanation of the meaning of the product of a line by a line shows a broadening of definition due to the Arab influence. Abraham solves certain equations, as $x^2 + b = ax$ (giving the double root), $x^2 + y^2 = a^2$, $x \pm y = b$, and $x^2 + y^2 = a^2$, $xy = b$, of course geometrically. The geometry, as was the custom of the time, due both to Arab and to Roman influence, is largely given to mensuration. The value of π is given as $3\frac{1}{4}$ or $377/120$ (that of Ptolemy), and the treatment of the circle is followed by some discussion of trigonometry, the table of chords being probably the oldest that is known in Latin form.

The second part of the volume is devoted to the correspondence of Regiomontanus with Giovanni Bianchini, Jacob von Speier and Christian Roder. This correspondence is preserved in the city library at Nürnberg, and although once published, §

* *Geschichte der Trigonometrie*, vol. 1, p. 93.

† *Zeitschrift für Mathematik und Physik*, XII, 1.

‡ See also his articles in the *Bibliotheca Mathematica*.

§ Christophori Theophili de Murr *Memorabilia Bibliothecarum Publicarum Norimbergensium et Universitatis Altorfinæ*. Pars 1., M.DCC.LXXXVI.

its reappearance is valuable both because it corrects certain errors in Murr's transcription, and because it now becomes generally accessible.

Regiomontanus was the leader of his generation in astronomy and mathematics, and his correspondence with Bianchini, who was court astronomer to the Duke of Ferrara, Speier, who was court astrologer to the Prince of Urbino, and Roder, the professor of mathematics at the University of Erfurt, throws much light upon the practical astronomical work of the fifteenth century. The correspondence is in Latin and no translation is given.

Altogether, this number of the *Abhandlungen* is one of the most valuable that have appeared, and the tendency to publish the sources for the history of mathematics is one that will meet the hearty commendation of scholars.

DAVID EUGENE SMITH.

Gauss' wissenschaftliches Tagebuch, 1796–1814. Mit Anmerkungen herausgegeben von FELIX KLEIN. Reprinted from the *Festschrift zur Feier des hundertfünfzigjährigen Bestehens der Königlichen Gesellschaft der Wissenschaften zu Göttingen*. Berlin, Weidmannsche Buchhandlung, 1901, 8vo., 44 pp.

As a youth not quite nineteen years old Gauss began jotting down in a copy-book memoranda, always, unfortunately, of the very briefest sort, of the great mathematical discoveries he was making. The entries in this Scientific Diary (*Catalogus*, Gauss calls it) are in Latin, and begin with a statement dated March 30, 1796, to the effect that Gauss had found a construction for the regular polygon of seventeen sides. From this date the entries follow each other in rapid succession, there being no less than 112 in the next four years and a quarter. From here on they become more irregular, and there are only 34 entries during the following fourteen years. Such a diary as this, written by any great mathematician, would be of the greatest interest, as illustrating, even with all its gaps and obscurities, the order in which the mathematical ideas developed in his mind and the form they first took; but there is probably no mathematician in whose case it could be even approximately as valuable as in the case of Gauss. For it is well known that ideas, many of them of the first importance, poured in on Gauss's mind in his early youth in such numbers that, as he himself

said, he was at times hardly able to master them; and many of these ideas, especially those concerning elliptic functions, were never incorporated in memoirs.

This diary has now been given to the public by Professor Klein, who has enhanced its value very greatly by attaching to a large number of its entries explanatory notes pointing out their relation to other published or unpublished writings of Gauss. Professor Klein explains, however, that this publication is to be regarded as a preliminary one only, since the diary is to be printed, with a more extended commentary, in Volume X, of Gauss's Collected Works.

A portrait of Gauss at the age of twenty-six, which has never before been published, serves as frontispiece, and one of the pages of the diary is reproduced in facsimile.

MAXIME BÔCHER.

Zur Integration partieller Differentialgleichungen. DR. KARL BOEHM. Leipzig, Teubner, 1900. 8vo. 55 pp. Mk. 1.80.

THE above paper, printed in book form, deals exclusively with the formal solution of partial differential equations, and of systems of such equations, by means of series, the convergence of which is not investigated.

The author starts with a given equation, or set of equations, of any order, forms their successive derived equations, and counts the number of derivatives of highest order compared with the number of equations, at each stage. He then shows, for a single equation, that the given equation can be solved with respect to any exceptional* derivative, and finally that a power series expansion may be obtained for the solution, which will formally satisfy the given equation, when arbitrary constant values have been preassigned for certain suitable derivatives; all this at a point where the equation is not "singular" with the respect to the exceptional derivative chosen. Similar results are obtained for a system of equations, under certain restrictions.

The work doubtless has real merit, but the reviewer does not feel justified in entering further into detail on account of the failure to discuss the convergence of the series in question: an omission which the author acknowledges in several places, and which renders the importance of the paper rather doubt-

* "Ausgezeichnet;" the definition is too intricate to repeat here.

ful. For it reduces the work to a questionable revision, not of Weierstrass and Kowalewsky, but of Cauchy, as the author confesses in his preface; certainly in fact as well as in principle not a modern tendency. In fact the author's only plea for a very complicated piece of reasoning, which might indeed have been put into far simpler form, is that "it gives results even for certain singular points, and fails only for the points which we have called *wholly singular*"—for several reasons a doubtful advantage, especially in view of the unfinished state of the work.

In conclusion, the material of the book would seem to be more fitted for a thesis; or for publication in a journal (in which case nearly twenty pages of repetition and unnecessary examples might profitably have been omitted); or for preservation unpublished, awaiting some of the proofs of convergence of which the author is happily sanguine. In its present form it may serve, if at all, perhaps to some student who will furnish the major portion of the whole by giving these convergence proofs. It is not in any sense a text-book or a treatise, and it is certainly not a book for general purchase.

E. R. HEDRICK.

The Measurement of General Exchange Value. BY CORREA MOYLAN WALSH. New York, The Macmillan Company, 1901. 8vo., xiv + 580 pp. \$3.00.

THIS is the most exhaustive work ever brought out on the theory of index numbers, embodies a vast deal of labor and acute logic, and will be a mine of information to future investigators. In so well worked ground there is necessarily much that is familiar to students of the subject, though hitherto inaccessible outside the largest libraries. But this editorial portion of the material—the most excellent bibliography, the index, the symbolized summaries of the methods of his predecessors—may prove to be the most valuable part of Mr. Walsh's work.

It is not in place in this BULLETIN to enter on the purely economic aspect of the subject, which is covered by an extended notice by Professor F. Y. Edgeworth in the *British Economic Journal* for September, 1901. But the mathematician and the practical statistician may find themselves puzzled by the confident rejection of the aid of the calculus of probabilities, elsewhere found so helpful both to theory and practice in dealing with complex problems of mensuration. Doubtless—as Professor

Edgeworth points out—the variations between two given series of index numbers according to different systems are within the range of probable error, and therefore inconclusive. What an amount of controversy might have been saved by general recognition of this fact.

To those interested in the theory of averages mention may be made of the appendices, in which are detailed the elementary propositions relating to the arithmetic, geometric and harmonic means, with single and multiple weighting.

The logical method of the work is admirable, its index most complete, but it leaves the reader with the impression that further work must be done and that no one solution will cover what is really a considerable number of independent problems.

J. M. GAINES.

NOTES.

THE closing (October) number of volume 3 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On the groups of order p^n which contain operators of order p^{n-2} ," by G. A. MILLER; "On the circuits of plane curves," by C. A. SCOTT; "Note on the real inflexions of plane curves," by C. A. SCOTT; "La théorie des plaques élastiques planes," by J. HADAMARD; "Covariants of systems of linear differential equations and applications to the theory of ruled surfaces," by E. J. WILCZYNSKI; "On the rank, order and class of algebraic minimum curves," by A. S. GALE; "On superosculating quadric surfaces," by H. MASCHKE; "Algebraic transformations of a complex variable realized by linkages," by A. EMCH; "On the determination of the distance between two points in space of m dimensions," by H. F. BLICHFELDT; "A definition of abstract groups," by E. H. MOORE; notes and errata: volumes 1, 2, 3.

THE October number (volume 24, number 4) of the *American Journal of Mathematics* contains: "On systems of linear differential equations of the first order," by M. BÔCHER; "On the quaternary linear homogeneous group and the ternary linear fractional group," by T. M. PUTNAM; "On cardinal numbers," by A. N. WHITEHEAD; "On a method of constructing all the groups of order p^n ," by G. A. MILLER; "Non-euclidean

properties of plane cubics and of their first and second polars," by H. F. STECKER.

At the meeting of the British association for the advancement of science held at Belfast, September 10-12, the president of Section A, Professor JOHN PURSER, took as the subject of his opening address an historical sketch of the Irish school of mathematics and physics. Sir NORMAN LOCKYER was elected president of the association for the coming year. The meeting in 1903 will be held at Southport, beginning September 9, and in 1904 at Cambridge. It is probable that the meeting in 1905 will be held at Cape Town.

THE several universities below offer during the winter semester of the current academic year courses in mathematics as follows :

UNIVERSITY OF GIESSEN.—By Professor M. PASCH : Differential calculus and the elements of integral calculus, four hours ; Introduction to the theory of functions, two hours ; Exercises in the elements of higher mathematics (algebra, analytic geometry, differential and integral calculus), two hours ; Exercises in the mathematical seminar, one hour.—By Professor E. NETTO : Definite integrals and their applications, four hours ; Analytic geometry of space, four hours ; Exercises in the mathematical seminar.

UNIVERSITY OF GREIFSWALD.—By Professor W. THOMÉ : Algebra, four hours ; Theory of plane algebraic curves, two hours ; Mathematical seminar, two hours.—By Professor E. STUDY : Non-euclidean geometry, two hours ; Mechanics, four hours ; Mathematical seminar, two hours.—By Professor G. KOWALEWSKI : Introduction to analytic geometry, two hours ; Theory of continuous transformation groups, two hours ; On Fourier's series (with applications to mathematical physics), two hours.

UNIVERSITY OF HALLE.—By Professor A. WANGERIN : Integral calculus, with exercises, five hours ; Calculus of variations, two hours ; Hydrodynamics, two hours ; Exercises in the mathematical seminar, one hour.—By Dr. H. GRASSMANN : Exercises in descriptive geometry, one hour ; Analytic geometry of space, two hours ; Elements of descriptive geometry, two hours.

UNIVERSITY OF KÖNIGSBERG.—By Professor F. MEYER : Exercises in integral calculus, one hour ; Integral calculus, four hours ; Differential geometry, four hours ; Exercises in differential geometry, one hour.—By Professor A. SCHOENFLIES : Mathematical seminar, two hours ; Applications of elliptic functions to geometry, mechanics and mathematical physics, four hours.—By Professor L. SAALSCHÜTZ : Exercises in algebraic analysis, one hour ; Theory of Fourier's series, two hours ; Introduction to algebraic analysis, two hours.

UNIVERSITY OF MARBURG.—By Professor F. SCHOTTKY : Algebraic analysis, four hours ; Mathematical seminar, two hours.—By Professor A. E. HESS : Integral calculus, five hours ; Spherical trigonometry and its applications, two hours ; Mathematical seminar, two hours.—By Dr. F. VON DALWIGK : Analytic geometry of space, especially surfaces of the second degree, four hours ; Analytic geometry and graphic statics, with exercises, two hours ; Advanced analytic geometry, one hour.—By Dr. H. JUNG : Algebra, four hours ; Differential and integral calculus, four hours.

UNIVERSITY OF MÜNSTER.—By Professor W. KILLING : Analytic mechanics, five hours ; Analytic geometry, five hours ; Theory of transformation groups, two hours ; Exercises in the mathematical proseminar, two hours ; Exercises in analytic geometry, one hour.—By Professor R. VON LILIENTHAL : Differential and integral calculus, five hours ; Introduction to the theory of differential equations, four hours ; Exercises in the mathematical seminar, one hour.—By Dr. M. DEHN : Elliptic functions, three hours ; Graphic statics, two hours ; Exercises in the theory of functions, one hour.

UNIVERSITY OF INNSBRUCK.—By Professor O. STOLZ : Theory of double integrals, two hours ; Elements of the calculus of variations, one hour ; Exercises in these subjects in the mathematical seminar, one hour ; Theory of functions of complex variables according to Cauchy and Weierstrass, three hours.—By Professor W. WIRTINGER : Higher algebra, three hours ; Abelian functions, two hours ; Mathematical seminar, two hours.—By Professor K. ZINDLER : Analytic geometry of the plane and of space, five hours ; Mathematical seminar, one hour.

UNIVERSITY OF NEUCHÂTEL.—By Professor L. ISELY : Infinitesimal calculus, including differential equations, three

hours ; Analytic geometry of space, two hours ; Theory of numbers, one hour.—By Professor L. GABEREL : General theory of functions, two hours ; Abelian functions, one hour.

UNIVERSITY OF PRAGUE.—By Professor G. PICK : Differential and integral calculus, five hours ; Mathematical seminar, two hours.—By Professor J. A. GMEINER : Analytic geometry, three hours ; Double integrals, two hours ; Exercises in analytic geometry, one hour.—By Dr. W. WEISS : Elements of descriptive geometry, two hours.

UNIVERSITY OF TÜBINGEN. — By Professor A. VON BRILL : Introduction to higher mathematics, four hours ; Non-rigid systems and the mechanics of Hertz, three hours ; Seminar, two hours.—By Professor H. STAHL : Higher algebra, three hours ; Applications of the theory of functions, four hours ; Seminar, two hours.—By Professor L. MAURER : Definite integrals, two hours ; Differential equations, two hours ; The theory of ternary forms, one hour, with exercises, one hour ; Exercises in higher analysis, two hours.

UNIVERSITY OF VIENNA.—By Professor G. R. VON ESCHERICH : Elements of the differential and integral calculus, with particular reference to their common applications, five hours ; Exercises relating to these lectures, one hour ; Proseminar, one hour ; Seminar, two hours.—By Professor L. GEGENBAUER : Integral calculus and calculus of variations, three hours ; Theory of spherical and cylindrical functions, with applications to problems of theoretical physics, two hours ; Exercises in the mathematical proseminar, one hour ; Exercises in the mathematical seminar, two hours.—By Professor F. MERTENS : Theory of numbers, five hours ; Exercises in the mathematical seminar, two hours ; Exercises in the mathematical proseminar, one hour.—By Professor G. KOHN : Introduction to synthetic geometry, four hours ; Exercises relating to these lectures, one hour ; Theory of invariants with applications to geometry, two hours.—By Dr. A. TAUBER : The mathematics of life insurance, four hours ; Exercises relating to this course, two hours.—By Dr. E. BLASCHKE : Introduction to mathematical statics, second course, three hours.—By Dr. R. DAUBLEBSKY VON STERNECK : Applications of differential and integral calculus to geometry, two hours ; Theory of numbers, one hour.—By Dr. K. CARDA : Introduction to the theory of contact transformations.—By Dr. R. SCHRAM : Method of least squares, one

hour.—By Professor H. HARTL: Elements of descriptive geometry, with applications in constructions, four hours; Geodesic coördinates, with computations, one and one half hours.

PROFESSOR F. SCHOTTKY, of Marburg, has been appointed to succeed the late Professor L. FUCHS at the University of Berlin.

DR. ALFRED LOEWY, of the University of Freiburg, has been promoted to an associate professorship of mathematics.

DR. H. HOHENNER of Munich has been appointed professor of geodesy in the Technical high school in Stuttgart.

PROFESSOR D. A. GRAVÉ of Kharkov has been called to a professorship of mathematics in the University of Kieff.

DR. E. HAENTZSCHEL has been appointed professor of mathematics in the Technical high school in Berlin.

PROFESSORS R. HAUSSNER, of Giessen, and A. KRAZER, of Strassburg, have been called to the Polytechnic school at Carlsruhe.

PROFESSOR L. M. DEFOE, of the department of mathematics of the Missouri State University, has been appointed professor in charge of the department of mechanics in that institution. Professor Defoe will enter upon his new duties on his return from Europe where he is spending a year on leave of absence.

DR. F. H. SAFFORD has been appointed instructor in mathematics at the University of Pennsylvania.

PROFESSOR W. H. METZLER, of Syracuse University, has been elected to membership in the Royal Society of Edinburgh.

AT Yale University, Mr. W. D. A. WESTFALL has been appointed instructor in mathematics in the academic department, and Mr. BURKE SMITH in the Sheffield scientific school. Dr. H. E. HAWKES has resumed his duties as instructor. Dr. E. B. WILSON is spending a year's leave of absence abroad.

JOHN WHELDON & Co., 36 Great Queen Street, London, W. C., have recently issued a catalogue (No. 14) of second-hand mathematical works, containing nearly 800 entries.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AHRENS (W.). See ENCYKLOPÄDIE.

BACHMANN (P.). *Niedere Zahlentheorie.* (In 2 Teilen.) Teil I. Leipzig, Teubner, 1902. 8vo. 10 + 402 pp. Cloth. (Teubner's Sammlung mathematischer Lehrbücher, Vol. X, 1.) M. 14.00

BOPP (K.). *Antoine Arnauld, der grosse Arnauld, als Mathematiker.* (Diss.) Heidelberg, 1902. 8vo. 49 pp.

CURTZE (M.). *Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance.* (In 2 Teilen.) Teil 2. Leipzig, Teubner, 1902. 8vo. Pp. 337-627. M. 14.00

DESSENON (E.). *Éléments de géométrie analytique, à l'usage des candidats à l'Ecole centrale et des élèves de première année de la classe de mathématiques spéciales.* 3e édition. Paris, Hachette, 1902. 8vo. 523 pp. (Cours complet d'études scientifiques.) Fr. 7.50

DOEHLEMANN (K.). *Geometrische Transformationen.* Teil 1: Die projektiven Transformationen nebst ihren Anwendungen. Leipzig, Göschen, 1902. 8vo. 7 + 322 pp. Cloth. (Sammlung Schubert, No. XXVII.) M. 10.00

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. (In 7 Bänden.) Vol. I: Arithmetik und Algebra, redigiert von W. F. Meyer. Heft 7: R. Mehmke, Numerisches Rechnen; W. Ahrens, Mathematische Spiele; V. Pareto, Anwendungen der Mathematik auf Nationalökonomie. Leipzig, Teubner, 1902. 8vo. Pp. 993-1120.

EYSANK (J. VON). *Einige Aufgaben aus der analytischen Geometrie.* (Progr.) Wien, 1902. 8vo. 25 pp.

FOUËT (E. A.). *Leçons élémentaires de la théorie des fonctions analytiques.* Partie I. Paris, 1902. 8vo. Pp. 1-310. Fr. 7.50

FRICKE (R.). *Hauptsätze der Differential- und Integralrechnung, als Leitfaden zum Gebrauch bei Vorlesungen zusammengestellt.* Teil 1. 2ter Abdruck. Braunschweig, Vieweg, 1902. 8vo. 9 + 80 pp. M. 2.00

FUCHS (R.). *Ueber lineare homogene Differentialgleichungen, deren Substitutionsgruppe von einem in den Koeffizienten auftretenden Parameter unabhängig ist.* (Progr.) Berlin, 1902. 4to. 23 pp.

FUHRMANN (W.). *Kollineare und orthologische Dreiecke.* (Progr.) Königsberg, 1902. 4to. 20 pp.

GANTER (H.) und RUDIO (F.). *Die Elemente der analytischen Geometrie, mit zahlreichen Übungsbeispielen.* Teil II: Die analytische Geometrie des Raumes, von F. Rudio. 3te, verbesserte Auflage. Leipzig, 1901. 8vo. 10 + 186 pp. Cloth. M. 3.00

- GROAT (B. F.). An introduction to the summation of differences of a function. An elementary exposition of the nature of the algebraic processes replaced by the abbreviations of the infinitesimal calculus. 4 + 43 pp. Seven lessons in theory of inversions of order and determinants. 32 pp. Minneapolis, Wilson, 1902. 8vo. Cloth.
- GRÜNWALD (A.). Geodätische Linien auf dem Ellipsoide. (Progr.) 1902. 8vo. 27 pp., 1 plate.
- HADAMARD (J.). La série de Taylor et son prolongement analytique. [Paris] Naud, 1901. 12mo. 10 + 102 pp. Cloth. (*Scientia*, série physico-mathématique, No. 12.) Fr. 2.00
- HALSTED (G. B.). The foundations of geometry. 8vo. (*The Open Court* 16, pp. 513-521.)
- HAUTH (R.). Ueber die Flächen, von deren Krümmungslinien ein System in parallelen Ebenen sich befindet. (Progr.) Metten, 1902. 8vo. 33 pp.
- HOYER. Andreas Gärtner, der "sächsische Archimedes." (Progr.) Dresden, 1902. 4to. 21 pp., 1 plate.
- KADESCH (A.). Ueber die Einhüllungsflächen von Potenzflächen-scharen. (Progr.) Wiesbaden, 1902. 4to. 45 pp.
- KINKELIN (H.). Quadraturen. (Progr.) Basel, Schwabe, 1902. 4to. 30 pp. M. 1.60
- KOCH (W.). Die Eigenschaften der Kurven 4ten Grades mit 2 Doppelpunkten, hergeleitet mittelst elliptischer Funktionen. (Progr.) Züllichau, 1902. 4to. 14 pp.
- LAURENT (H.). Sur les principes fondamentaux de la théorie des nombres et de la géométrie. [Paris] Naud, 1902. 12mo. 68 pp. Cloth. (*Scientia*, série physico-mathématique, No. 20.) Fr. 2.00
- LODGE (A.). Differential calculus for beginners. With an introduction by Sir O. J. Lodge. London, Bell, 1902. 12mo. 304 pp. Cloth. 4s. 6d.
- MAUPIN (G.). Opinions et curiosités touchant la mathématique. 2e série. Paris, Naud, 1902. 8vo. 338 pp.
- MEHMKE (R.), MEYER (W. F.). See ENCYKLOPÄDIE.
- MIORINI (W. VON). Ueber eine Erweiterung der Sätze von Pascal und Brianchon. (Progr.) Wien, 1902. 8vo. 12 pp., 1 plate.
- MORITZ (R. E.). Generalization of the differentiation process. 4to. (*American Journal of Mathematics* 24, pp. 257-301.)
- PARETO (V.). See ENCYKLOPÄDIE.
- PASCAL (E.). Lezioni di calcolo infinitesimale. Parte I: Calcolo differenziale. 2a edizione completamente riveduta. Milano, Hoepli, 1902. 16mo. 12 + 331 pp. (Manuali Hoepli.)
- PILGRAM (A.). Die Scharschar der Kegelschnitte, die ein gegebenes Tangendendreieck haben. (Diss.) Erlangen, 1902. 8vo. 27 pp., 1 plate.
- RUDIO (F.). See GANTER (H.).
- SCHUBERT (H.). Niedere Analysis. Teil I: Kombinatorik, Wahrscheinlichkeitsrechnung, Kettenbrüche und diophantische Gleichungen. Leipzig, Göschen, 1902. 8vo. 7 + 181 pp. Cloth. (Sammlung Schubert, No. V.) M. 3.60

- VERONESE (G.). Les postulats de la géométrie dans l'enseignement. 8vo. (Deuxième Congrès international des mathématiciens, Paris, 1900, pp. 433-450.)
- VILLANI (N.). Ricerche matematiche sulle misure antiche e il sistema antico delle misure romane. Lanciano, Carabba, 1902. 16mo. 7 + 66 pp. Fr. 1.00
- ZINDLER (K.). Liniengeometrie mit Anwendungen. Vol. I. Leipzig, Götschen, 1902. 8vo. 8 + 380 pp. Cloth. (Sammlung Schubert, No. XXXIV.) M. 12.00
- ZÜHLKE (P.). Ueber die geodätischen Linien und Dreiecke auf den Flächen konstanten Krümmungsmasses und ihre Beziehungen zur sogenannten nicht-Euklidischen Geometrie. (Progr.) Charlottenburg, 1902. 8vo. 36 pp., 1 plate.

II. ELEMENTARY MATHEMATICS.

- ASCARZA (V. J.) y SOLANA (E.). Colección de problemas de aritmética razonados y resueltos analíticamente; aprobado de real orden para servir de texto. 3a edición corregida y aumentada. Madrid, Imp. Moderna, 1902. 8vo. 183 pp. Fr. 1.50
- BROOKS (E.). Plane and solid geometry; a complete course in the elements of the science. Revised edition. Philadelphia, Christopher Sower Co. [1902]. 12mo. 415 pp. Cloth. \$1.25
- CHAILAN (E.). Géométrie, à l'usage des élèves des classes de lettres, conforme au programme de la première partie du baccalauréat de l'enseignement secondaire classique. 2e édition. Paris, Pousielgue, 1902. 16mo. 8 + 240 pp. (Alliance des maisons d'éducation chrétienne.)
- CHANCELLOR (W. E.). Elementary school mathematics by grades. Seventh book: arithmetic, geometry, and algebra. New York, Globe School Book Co. [1902]. 12mo. 176 pp. Cloth. (Globe series.) \$0.28
- DAVISON (C.). Easy mathematical problem papers; with answers. London, Blackie, 1902. 12mo. 12 pp. Cloth. 2s. 6d.
- FLETCHER (W. C.). Elementary geometry. London, Arnold, 1902. 12mo. 84 pp. Cloth. 1s. 6d.
- FÜCHTJOHANN (H.). Lösung der Aufgaben in J. R. Boyman's Lehrbuch der Planimetrie. 2ter Teil: Aufgabe 734-1244. Bonn, Cohen, 1902. 8vo. 212 pp. M. 4.20
- GIKA (D.) and MUBOMTSEV (A.). Geometrical problems. Part II: Solid geometry. Moscow, 1902. 8vo. 163 pp. (Russian.) M. 3.00
- GODT (W.). Ueber einige sogenannte merkwürdige Punkte des Dreiecks. I. (Progr.) Lübeck, 1902. 4to. 23 pp.
- HOPKINS (G. I.). Inductive plane geometry; with numerous exercises, theorems, and problems for advance work. New revised edition. Boston, Heath, 1902. 12mo. 6 + 208 pp. Hf. leather. \$0.75

- JELINEK (L.).** Mathematische Tafeln für technische Lehranstalten, besonders für höhere Gewerbeschulen. 3te Auflage; nach der neuen Rechtschreibung berichtiger, sonst im wesentlichen unveränderter Abdruck der 2ten Auflage. Wien, Pichler, 1902. 8vo. 223 pp. Cloth. M. 2.00
- LORENZ (K.).** Das Rechnen mit unvollständigen Dezimalbrüchen. (Progr.) Weidenhofen, 1902. 8vo. 22 pp.
- LÖSUNGEN** der Absolutorial-Aufgaben aus der Mathematik an den humanistischen Gymnasien Bayerns. 8ter Nachtrag: 1902. München, Pohl, 1902. 8vo. 7 pp. M. 0.30
- MAHLER (G.).** Ebene Geometrie. 3te, verbesserte Auflage, 2ter Abdruck. Leipzig, Göschen, 1902. 12mo. 158 pp. Cloth. (Sammlung Göschen, No. 41.) M. 0.80
- MUROMTSEV (A.).** See GIKA (D.).
- PETERS (F.).** Ziegler's graphische Darstellung der trigonometrischen Funktionen nebst Tafeln zur Konstruktion bestimmter Winkel und Linien. Ein praktisches Hilfsmittel beim geometrischen Zeichnen. Wiesbaden, Kreidel, 1902. 8vo. 22 pp., 6 plates. Cloth. M. 3.00
- SÁNCHEZ CASADO (F.) y SÁNCHEZ RUEDA (E.).** Prontuario de geometría y trigonometría, para uso de los alumnos de los institutos, colegios y seminarios. 7a edición corregida y notablemente aumentada por E. Sánchez Rueda. Madrid, Hernando y de Jubera, 1902. 8vo. 127 pp. Fr. 1.00
- SOLANA (E.).** See ASCARZA (V. J.).

III. APPLIED MATHEMATICS.

- APARICI (R.).** Lecciones de geometría descriptiva. 2a edición. (Rectas y planos, poliedros, superficies cilíndricas, cónicas y de revolución; planos acotados.) Madrid, Lib. Gutenberg de José Ruiz, 1903. 4to. 8 + 32 + 8 pp. Fr. 8.00
- APPELL (P.).** Traité de mécanique rationnelle (cours de mécanique de la Faculté des sciences de Paris). Vol. III: Equilibre et mouvement des milieux continus. 3e fascicule. Paris, Gauthier-Villars, 1903. 8vo. Pp. 457-558.
- ATKINSON (A. A.).** Electrical and magnetic calculations. New York, 1902. 8vo. 7 + 310 pp. Cloth. \$1.50
- BRÜSCH (W.).** Grundriss der Elektrotechnik für technische Lehranstalten. Leipzig, Teubner, 1902. 8vo. 11 + 168 pp. Cloth. M. 3.00
- BURZHEIEV (S.).** The elements of the resistance of materials and graphostatics, with applications to the construction of machines. St. Petersburg, 1902. 8vo. 207 pp. (Russian.) R. 1.50
- CESÀRO (E.).** Sur un problème de propagation de la chaleur. 8vo. (*Bulletin de l'Académie royale de Belgique, classe des sciences*, 1902, pp. 387-404.)
- CIURLO (M.).** Sul principio dei lavori virtuali; memoria. Genova, Istituto Sordomuti, 1902. 8vo. 9 pp.

- FITZGERALD (G. F.). Scientific writings, collected and edited, with a historical introduction by J. Larmor. London, Longmans, 1902. 8vo. Portrait. Cloth. 15s.
- GENOVINO (G.). La variabilità degli angoli che la congiungente le punte del crescente lunare forma col circolo di declinazione e col verticale del centro della luna e problemi relativi. Pistoia, Niccolai, 1902. 8vo. 36 pp.
- . L'epoca della massima visibilità del pianeta Venere. Firenze, Landi, 1902. 8vo. 13 pp.
- . Le epoche in cui la luna in una sua rivoluzione intorno alla terra si trova: 1° alla minima distanza da due stelle; 2° ad ugual distanza da due stelle; 3° sul circolo massimo che passa per due stelle. Pistoia, Niccolai, 1902. 8vo. 13 pp.
- HARTMANN (R.). Beiträge zur Wirbelbewegung. (Diss.) Braunschweig, 1902. 8vo. 31 pp.
- HAUG (J.). Ueber die Drehung eines starren Körpers um seinen Schwerpunkt. (Progr.) München, 1902. 8vo. 17 pp.
- HEUN (K.). Formeln und Lehrsätze der allgemeinen Mechanik, in systematischer und geschichtlicher Entwicklung dargestellt. Leipzig, Göschen, 1902. 8vo. 8 + 112 pp. Cloth. M. 3.50
- HOFF (J. H. VAN'T). Acht Vorträge über physikalische Chemie, gehalten auf Einladung der Universität Chicago. Braunschweig, Vieweg, 1902. 8vo. 7 + 81 pp. M. 2.50
- JOHNSON (L. J.). The determination of unit stresses in the general case of flexure. 8vo. 39 pp. (*Journal of the Association of Engineering Societies* 27, 1892.)
- KLOSS (M.). Analytisch-graphisches Verfahren zur Bestimmung der Durchbiegung zwei- und dreifach gestützter Träger. Mit besonderer Berücksichtigung von Drehstrommotorenwellen. (Diss.) Berlin, Polytechnische Buchhandlung, 1902. 8vo. 128 pp., 4 plates. M. 3.00
- KOTTENBACH (R.). Zur didaktischen Behandlung einiger Fragen der Mechanik. (Progr.) Troppau, 1902. 8vo. 28 pp.
- KRIEMLER (C. J.). Labile und stabile Gleichgewichtsfiguren vollkommen elastischer auf Biegung beanspruchter Stäbe mit besonderer Berücksichtigung der Knickvorgänge. (Habilitationsschrift.) Karlsruhe, Braun, 1902. 4to. 56 pp., 10 plates.
- LANGER (K.). Direkte Konstruktion der Konturen von Rotationsflächen II. Ordnung in orthogonaler Darstellung. (Progr.) Horn, 1902. 8vo. 12 pp.
- LARMOR (J.). See FITZGERALD (G. F.).
- OVAZZA (E.). Urti ed esplosioni; lezioni di dinamica applicata. Torino, Lattes, 1902. 8vo. 155 pp. Fr. 5.00
- PETRUS (A.). Beiträge zur Theorie der Herpolhodie Poinso's. (Diss.) Halle, 1902.
- POEZL (W.). Anfangsgründe der darstellenden Geometrie, enthaltend die geradlinigen ebenen Gebilde, zum Schulgebrauche zusammengestellt. 2te Auflage. München, Ackermann, 1902. 8vo. 4 + 58 pp. M. 1.20

- . Elemente der darstellenden Geometrie, für höhere Lehranstalten zusammengestellt. 2 Teile. 2te Auflage. Teil 1: Geradlinige Gebilde. 6 + 112 pp. Teil 2: Krümmflächige Gebilde. 6 + 111 pp. München, Ackermann, 1902. 8vo. M. 4.00
- QUESNEVILLE (G.). Nouvelle dioptrique des rayons visuels; théorie nouvelle de la lunette de Galilée. Paris, Hermann, 1902. 8vo. 64 pp.
- RÜLF (B.). Der Reguliervorgang bei Dampfmaschinen. (Diss.) Berlin, 1902. 4to. 15 pp.
- SANTEL (A.). Bemerkungen zur Didaktik einiger Kapitel der Mechanik. (Progr.) Götz, 1902. 8vo. 41 pp.
- SCHEEFFER (E.). Ueber stabiles Schwimmen homogener Körper. (Progr.) Danzig, 1902. 4to. 52 pp.
- SPIRGATIS (E.). Berechnung der Fahrzeiten aus den Zugkräften der Dampflokomotiven. Leipzig, Spigatis, 1902. 8vo. 68 pp. M. 3.50
- STOLLE (R.). See WEICKERT (A.).
- TÄGERT (W.). Ueber Schwankungen der Drehungsachse der Erde im Innern des Erdkörpers. (Progr.) Siegen, 1902. 4to. 30 pp.
- WEBER (R. H.). Elektromagnetische Schwingungen in Metallröhren. (Habilitationsschrift.) Heidelberg, 1902. 8vo. 33 pp.
- WEICKERT (A.) und STOLLE (R.). Praktisches Maschinenrechnen. 5te Auflage (9tes und 10tes Tausend). Berlin, Polytechnische Buchhandlung, 1902. 8vo. 7 + 292 pp. Cloth. M. 5.00

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The following dates have been fixed for the meetings of the Society:

	A. M.		A. M.
Sat.,	Oct. 25, 1902, 11:00	Sat.,	Feb. 28, 1903, 11:00
Mo.-Tu.,	Dec. 29-30, " "	"	April 25, " "

The Meeting of December 29-30 will be the Annual Meeting of the Society for the Election of Officers and Delivery of the Presidential Address.

BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY



CONTINUATION OF THE BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

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CONCERNING THE COMMUTATOR SUBGROUPS OF GROUPS WHOSE ORDERS ARE POWERS OF PRIMES.

BY DR. W. B. FITE.

(Read before the American Mathematical Society, September 2, 1902.)

IN the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY, volume 3 (1902), pages 331, 351, the writer has shown that the commutators of a metabelian group are invariant in the group, and that those operators of a group which correspond to the invariant operators of the group of cogredient isomorphisms are commutative with all the commutators of the group. It follows as a direct consequence of these two facts that the commutator subgroup of a group of the second or third class is abelian, that is, is of the first class. It was shown in the same article, page 349, that a metabelian group of odd order cannot be a group of cogredient isomorphisms if it has a set of generators such that the order of any one of them is not a divisor of the least common multiple of all the others. It is the main object of this paper to show that these facts are special cases of more general ones.

Let G be a group of order p^m (p being a prime) and let G' be the group of cogredient isomorphisms of G , and G'' that of G' ; also let l and l'' denote the classes of the commutator subgroups of G and G'' respectively. If B_1 is any commutator of G , and A_i ($i = 1, 2, \dots, l''$) a set of any l'' commutators (not necessarily distinct) of G , we have $A_i^{-1} B_1 A_i = B_1 B_{i+1}$. Now since the commutator subgroup of G'' is of class l'' it is evident that $B_{i''}$ is invariant in this subgroup ($B_{i''}$ being that operator of G'' that corresponds to $B_{i''}$). Therefore $B_{i''+1}$ is invariant in the commutator subgroup of G and $l \leq l'' + 1$. We have seen that for groups of classes two or three the commutator subgroups are of class one. We have therefore proved the

THEOREM: *If a group G is of order p^m (p being a prime) and class $2k$ or $2k+1$, its commutator subgroup is of class l , where $l \leq k$.*

Suppose now that G is of class k , where $k \leq p$, and has an abelian commutator subgroup. Let A_i ($i = 1, 2, \dots, n$), be a set of generators of G' of orders p^{a_i} respectively, and let A_i be

an operator of G that corresponds to A'_i . If B is any operator of G , we have $A_i^{-1}BA_i = Bt_l$, and

$$A_i^{-1}t_jA_i = t_{j+1} \quad (j=1, 2, 3, \dots, k-1).$$

It follows from this that

$$A_i^{-p^{a_i}}BA_i^{p^{a_i}} = Bt_1^{p^{a_i}}t_2^{\frac{p^{a_i}(p^{a_i}-1)}{2!}} \dots t_j^{\frac{p^{a_i}(p^{a_i}-1)}{j!} \dots (p^{a_i}-j+1)} \dots$$

$$t_{k-1}^{\frac{p^{a_i}(p^{a_i}-1)}{(k-1)!} \dots (p^{a_i}-k+2)} = B,$$

since A'_i is of order p^{a_i} . Also

$$A_i^{-p^{a_i}}t_jA_i^{p^{a_i}} = t_{j+1}^{p^{a_i}}t_{j+2}^{\frac{p^{a_i}(p^{a_i}-1)}{2}} \dots t_{k-1}^{\frac{p^{a_i}(p^{a_i}-1)}{(k-j-1)!} \dots (p^{a_i}-k+j+2)} = t_j,$$

$$(j=1, 2, \dots, k-2)$$

and $t_{j+1}^{p^{a_i}} = 1$, provided $t_l^{p^{a_i}} = 1$ ($l > j+1$). Now $t_{k-1}^{p^{a_i}} = 1$, and therefore $t_1^{p^{a_i}} = 1$.

In a similar way we can show that the order of t_1 is a divisor of the order of B' . Also the order of t_{j+1} ($j=1, 2, \dots, k-2$) is a divisor of the order of t'_j and therefore a divisor of the order of B' . It follows from this that if an operator B corresponds to an operator of G' of order p^β , then it is commutative with the p^β th power of every operator of G .

Suppose now that $\alpha_1 > \alpha_i$ ($i \geq 2$). Then a power of A_1 , less than p^{α_1} , would be commutative with every operator of G . But this is impossible.

THEOREM: *If a group G , of order p^m (p being a prime) and class k ($k \leq p$), has an abelian commutator subgroup, G' cannot have a set of generators such that the order of one of them is greater than the order of every one of the others.*

We now consider under what conditions a group G of order $p_1^{a_1}p_2^{a_2} \dots p_n^{a_n}$, where $p_1, p_2, \dots, p_n$ are distinct primes, can have a commutator subgroup of order $p_1^{a_1}$.* Let $G_1, G_2, \dots, G_n$ be subgroups of G of orders $p_1^{a_1}, p_2^{a_2}, \dots, p_n^{a_n}$ respectively. Then G_i ($i > 1$) is abelian, since identity is the only commutator it contains. Moreover, any operator of G that transforms G_i into

* The symmetric group of degree three is such a group. It is of order 3.2 and its commutator subgroup is of order 3.

itself is commutative with every operator of G_i . Let H_1 be the commutator subgroup. The group $\{H_1, G_i\}$ is of order $p_i^\beta p_i^{\alpha_i}$. This contains $p_i^\gamma (\gamma \leq \beta)$ subgroups of order $p_i^{\alpha_i}$, and therefore $p_i^\gamma \equiv 1 \pmod{p_i}$. Hence if

$$p_i^\gamma \not\equiv 1 \pmod{p_i} \quad (0 < \gamma \leq \beta),$$

every commutator is commutative with every operator of G_i . Then $A_j^{-1} A_i A_j = A_i t_i$, where A_j is any operator of

$$G_j \quad (j = 1, 2, \dots, n)$$

and A_i is any operator of G_i ; and $A_j^{-1} A_i^{\beta_i} A_j = A_i^{\beta_i}$, where $p_i^{\beta_i}$ is the order of t_i . But $p_i^{\beta_i}$ is relatively prime to p_i . Therefore $A_j^{-1} A_i A_j = A_i$, and G is the direct product of the groups G_j .

THEOREM. *If a group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ ($p_1, p_2, \dots, p_n$ being distinct primes) has a commutator subgroup of order p_1^β and if $p_i^\gamma \not\equiv 1 \pmod{p_i}$ ($0 < \gamma \leq \beta$), ($i = 2, 3, \dots, n$), then G is the direct product of groups of orders $p_1^{\alpha_1}, p_2^{\alpha_2}, \dots, p_n^{\alpha_n}$ respectively.*

CORNELL UNIVERSITY.

August, 1902.

NOTE ON IRREGULAR DETERMINANTS.

BY PROFESSOR L. I. HEWES.

IN Gauss's table * of binary quadratic forms the two negative determinants — 468 and — 931 of the first thousand are classed as regular and their genera and classes given correctly. Perott † has pointed out that these two determinants are irregular. The details of the classes of the original thirteen irregular determinants of Gauss have been worked out by Cayley ‡ and on the following page are given the details, in his notation, for the properly primitive reduced forms of the two determinants added by Perott's investigation.

* C. F. Gauss, Werke, vol. II, p. 450.

† "Sur la formation des déterminants irréguliers," *Crelle*, vol. 59.

‡ Cayley's Collected Papers, vol. 5, p. 141.

D	Classes.			α	β	δ	Cp.
— 468	1	0	468	+	+	+	1
	13	0	36				e^2
	9	0	52				e_1^2
	4	0	117				$e^2 e_1^2$
	9	3	53	—	+	+	e_1
	17	— 5	29				$e^2 e_1$
	17	5	29				$e^2 e_1^3$
	9	— 3	53				e_1^3
	19	8	28	+	—	—	e
	7	— 1	67				$e^2 e_1^2$
	7	1	67				$e e_1^2$
	19	— 8	28				e^3
	8	2	59	—	—	—	$e^3 e_1$
	11	— 4	44				$e e_1^3$
	11	4	44				$e^3 e_1^3$
	8	— 2	59				$e e_1^3$
							Exponent of irregularity 2.

D	Classes.			α	β	Cp.			
— 931	1	0	931	+	+	1			
	4	1	233			g^4			
	4	— 1	233			g^2			
	25	12	43			dg^4			
	25	— 12	43			d^2g^2			
	11	2	85			d			
	11	— 2	85			d^2			
	23	9	44			dg^2			
	23	— 9	44			d^2g^4			
	19	0	49	+	—	g^3			
	17	2	55			g			
	17	— 2	55			g^5			
	5	— 2	187			dg^5			
	5	2	187			d^2g^5			
	20	3	47			dg^3			
	20	— 3	47			d^2g^3			
	20	7	49			d^2g			
	20	— 7	49			dg^5			
							Exponent of irregularity 3.		

KINGSTON, R. I.,
August, 1902.

NOTE ON THE PROJECTIONS OF THE ABSOLUTE ACCELERATION IN RELATIVE MOTION.

BY DR. G. O. JAMES.

INASMUCH as the usual methods of forming the projections of the absolute acceleration on the moving axes rest to a certain extent on geometric considerations it has seemed worth while to treat the question from a purely analytic point of view and derive these expressions by purely algebraic processes. This method possesses the advantage of being immediately extensible to any number of dimensions, and in the latter part of this note I shall give the expressions for n dimensions.

Connecting the moving axes Ox , Oy , Oz with the fixed axes by the following schema

$$\begin{array}{c|ccc} & x & y & z \\ X & a & b & c \\ Y & a' & b' & c' \\ Z & a'' & b'' & c'' \end{array}$$

I shall rapidly recall the following equations :

$$(1) \quad X = Sax, \quad Y = Sa'x, \quad Z = Sa''x,$$

where S denotes summation with respect to a , b , c and x , y , z .

$$(2) \quad \begin{cases} \frac{dX}{dt} = Sa \frac{dx}{dt} + Sx \frac{da}{dt}, \\ \frac{dY}{dt} = Sa' \frac{dx}{dt} + Sx \frac{da'}{dt}, \\ \frac{dZ}{dt} = Sa'' \frac{dx}{dt} + Sx \frac{da''}{dt}; \end{cases}$$

$$(3) \quad \begin{cases} \frac{d^2 X}{dt^2} = S a \frac{d^2 x}{dt^2} + S x \frac{d^2 a}{dt^2} + 2 S \frac{dx}{dt} \frac{da}{dt}, \\ \frac{d^2 Y}{dt^2} = S a' \frac{d^2 x}{dt^2} + S x \frac{d^2 a'}{dt^2} + 2 S \frac{dx}{dt} \frac{da'}{dt}, \\ \frac{d^2 Z}{dt^2} = S a'' \frac{d^2 x}{dt^2} + S x \frac{d^2 a''}{dt^2} + 2 S \frac{dx}{dt} \frac{da''}{dt}; \end{cases}$$

$$(4) \quad \begin{cases} p = \sum^c b \frac{db}{dt} = - \sum^b b \frac{dc}{dt}, \\ q = \sum^a b \frac{dc}{dt} = - \sum^c b \frac{da}{dt}, \\ r = \sum^b a \frac{da}{dt} = - \sum^a a \frac{db}{dt}. \end{cases}$$

The projections of the absolute velocity on the moving axes are then

$$(4) \quad \begin{cases} V_x = \sum^a a \frac{dX}{dt} = \frac{dx}{dt} + qz - ry, \\ V_y = \sum^b b \frac{dX}{dt} = \frac{dy}{dt} + rx - pz, \\ V_z = \sum^c c \frac{dX}{dt} = \frac{dz}{dt} + py - qx. \end{cases}$$

The direction cosines satisfy the following equations

$$(5) \quad \frac{da}{dt} = br - cq, \quad \frac{db}{dt} = cp - ar, \quad \frac{dc}{dt} = aq - bp,$$

with two similar sets in a', b', c' , and a'', b'', c'' , which I shall denote by (5)' and (5)''.

The projections of the absolute acceleration on the moving axes are

$$(6) \quad \begin{cases} J_x = \sum^a a \frac{d^2 X}{dt^2} = \sum^a S a \frac{d^2 x}{dt^2} + \sum^a S x \frac{d^2 a}{dt^2} + 2 \sum^a a S \frac{dx}{dt} \frac{da}{dt}, \\ J_y = \sum^b b \frac{d^2 X}{dt^2} = \dots, \\ J_z = \sum^c c \frac{d^2 X}{dt^2} = \dots. \end{cases}$$

Calculating $\sum aS\alpha \frac{d^2x}{dt^2}$ and remembering the relations between the direction cosines we have at once

$$(7) \quad \sum aS\alpha \frac{d^2x}{dt^2} = \frac{d^2x}{dt^2}.$$

Again

$$\sum aSx \frac{d^2a}{dt^2} = x \sum a \frac{d^2a}{dt^2} + y \sum a \frac{d^2b}{dt^2} + z \sum a \frac{d^2c}{dt^2}.$$

Differentiating (5) (5)' and (5)'' and calculating

$$\sum a \frac{d^2a}{dt^2}, \quad \sum a \frac{d^2b}{dt^2}, \quad \sum a \frac{d^2c}{dt^2},$$

we have

$$(8) \quad \begin{cases} \sum a \frac{d^2a}{dt^2} = -(q^2 + r^2), \\ \sum a \frac{d^2b}{dt^2} = -\frac{dr}{dt} + pq, \\ \sum a \frac{d^2c}{dt^2} = \frac{dq}{dt} + pr. \end{cases}$$

Again

$$(9) \quad \begin{aligned} \sum aS \frac{dx}{dt} \frac{da}{dt} &= \frac{dx}{dt} \sum a \frac{da}{dt} + \frac{dy}{dt} \sum a \frac{db}{dt} + \frac{dz}{dt} \sum a \frac{dc}{dt} \\ &= -r \frac{dy}{dt} + q \frac{dz}{dt}. \end{aligned}$$

Hence

$$(10) \quad \begin{aligned} J_x &= \frac{d^2x}{dt^2} - (q^2 + r^2)x + \left(pq - \frac{dr}{dt}\right)y + \left(pr + \frac{dq}{dt}\right)z \\ &\quad - 12\left(r \frac{dy}{dt} - q \frac{dz}{dt}\right).^* \end{aligned}$$

Now (10) may be written

$$(11) \quad \begin{aligned} J_x &= \frac{d}{dt} \left(\frac{dx}{dt} + qz - ry \right) + q \left(\frac{dz}{dt} + py - qx \right) \\ &\quad - r \left(\frac{dy}{dt} + rz - pz \right). \end{aligned}$$

* This formula appears to have been first given by J. Liouville, *Jour. de Math. pures et appl.*, series 2, vol. 3 (1858), pp. 10-11.

Hence we have for the projections the ordinary expressions

$$(11) \quad \begin{cases} J_x = \frac{dV_x}{dt} + qV_z - rV_y, \\ J_y = \frac{dV_y}{dt} + rV_x - pV_z, \\ J_z = \frac{dV_z}{dt} + pV_y - qV_x. \end{cases}$$

To extend this method to n dimensions I shall connect the moving axes $Ox_1, Ox_2, \dots, Ox_n$ with the fixed axes $OX_a, OX_\beta, \dots, OX_\nu$ by the following table :

	x_1	x_2	$\dots$	x_n
X_a	a_1	a_2	$\dots$	a_n
X_β	β_1	β_2	$\dots$	β_n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_ν	ν_1	ν_2	$\dots$	ν_n

Then

$$(12) \quad X_\lambda = \sum_{i=1}^n \lambda_i x_i,$$

$$(13) \quad \frac{dX_\lambda}{dt} = \sum_{i=1}^n \lambda_i \frac{dx_i}{dt} + \sum_{i=1}^n x_i \frac{d\lambda_i}{dt},$$

$$(14) \quad \frac{d^2 X_\lambda}{dt^2} = \sum_{i=1}^n \lambda_i \frac{d^2 x_i}{dt^2} + \sum_{i=1}^n x_i \frac{d^2 \lambda_i}{dt^2} + 2 \sum_{i=1}^n \frac{dx_i}{dt} \frac{d\lambda_i}{dt},$$

$$(15) \quad \begin{aligned} V_{x_r} &= S a_r \frac{dX_a}{dt} = S a_r \sum_{i=1}^n a_i \frac{dx_i}{dt} + S a_r \sum_{i=1}^n x_i \frac{da_i}{dt} \\ &= \sum_{i=1}^n S a_r a_i \frac{dx_i}{dt} + \sum_{i=1}^n S a_r \frac{da_i}{dt} x_i \\ &= \frac{dx_r}{dt} + \sum_{i=1}^n x_i S a_r \frac{da_i}{dt} = \frac{dx_r}{dt} + \sum_{i=1}^n x_i p_{ir}, \end{aligned}$$

where

$$(16) \quad p_{i\tau} \equiv Sa_{\tau} \frac{da_i}{dt} = -Sa_i \frac{da_{\tau}}{dt},$$

$$(17) \quad J_{x_{\tau}} = Sa_{\tau} \frac{d^2 X_a}{dt^2} = Sa_{\tau} \sum_{i=1}^n a_i \frac{d^2 x_i}{dt^2} + Sa_{\tau} \sum_{i=1}^n x_i \frac{d^2 a_i}{dt^2} \\ + 2Sa_{\tau} \sum_{i=1}^n \frac{dx_i}{dt} \frac{da_i}{dt} \\ = \sum_{i=1}^n \frac{d^2 x_i}{dt^2} Sa_{\tau} a_i + \sum_{i=1}^n x_i Sa_{\tau} \frac{d^2 a_i}{dt^2} + 2 \sum_{i=1}^n \frac{dx_i}{dt} Sa_{\tau} \frac{da_i}{dt}.$$

The differential equation satisfied by the direction cosine λ_i is found by expressing the fact that the point at unit distance on the axis OX_{λ} has a velocity zero along the moving axes. Hence

$$(18) \quad \frac{d\lambda_i}{dt} + \sum_{j=1}^n \lambda_j p_{ji} = 0.$$

From (18)

$$(19) \quad \frac{d^2 \lambda_i}{dt^2} = \sum_{j=1}^n \lambda_j \frac{dp_{ij}}{dt} + \sum_{j=1}^n \frac{d\lambda_j}{dt} p_{ij}.$$

Whence

$$(20) \quad Sa_{\tau} \frac{d^2 a_i}{dt^2} = \sum_{j=1}^n \frac{dp_{ij}}{dt} Sa_{\tau} a_j + \sum_{j=1}^n p_{ij} Sa_{\tau} \frac{da_j}{dt} = \frac{dp_{i\tau}}{dt} + \sum_{j=1}^n p_{ij} p_{j\tau}.$$

Hence

$$(21) \quad J_{x_{\tau}} = \frac{d^2 x_{\tau}}{dt^2} + \sum_{i=1}^n x_i \left(\frac{dp_{i\tau}}{dt} + \sum_{j=1}^n p_{ij} p_{j\tau} \right) + 2 \sum_{i=1}^n \frac{dx_i}{dt} p_{i\tau} \\ = \frac{d}{dt} \left(\frac{dx_{\tau}}{dt} + \sum_{i=1}^n x_i p_{i\tau} \right) + \sum_{i=1}^n \left(\frac{dx_i}{dt} + \sum_{j=1}^n x_j p_{ji} \right) p_{i\tau} \\ = \frac{d}{dt} V_{x_{\tau}} + \sum_{i=1}^n V_{x_{\tau}} p_{i\tau} \quad (\tau = 1, 2 \dots, n).$$

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INFINITESIMAL DEFORMATION OF THE SKEW HELICOID.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, September 3, 1902.)

CONSIDER the skew helicoid S , defined by the equations

$$(1) \quad x = u \cos v, \quad y = u \sin v, \quad z = av.$$

We shall show that the problem of the infinitesimal deformation of this surface can be completely solved.

By direct calculation we find

$$(2) \quad E = \sum \left(\frac{\partial x}{\partial u} \right)^2 = 1, \quad F = \sum \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} = 0,$$

$$G = \sum \left(\frac{\partial x}{\partial v} \right)^2 = u^2 + a^2,$$

and

$$(3) \quad X, Y, Z = \frac{a \sin v, -a \cos v, u}{\sqrt{u^2 + a^2}}$$

where Y, X, Z denote the direction cosines of the normal. Again we find

$$(4) \quad D = \sum X \frac{\partial^2 x}{\partial u^2} = 0, \quad D' = \sum X \frac{\partial^2 x}{\partial u \partial v} = \frac{-a}{\sqrt{u^2 + a^2}},$$

$$D'' = \sum X \frac{\partial^2 x}{\partial v^2} = 0.$$

The characteristic equation of the deformation reduces in this case to

$$\frac{\partial^2 \phi}{\partial u \partial v} + \frac{u}{u^2 + a^2} \frac{\partial \phi}{\partial v} = 0,$$

of which the general integral is

$$(5) \quad \phi = \frac{U + V}{\sqrt{u^2 + a^2}},$$

where U is a function of u alone and V is a function of v alone.

The cartesian cōordinates of the surface S_1 , corresponding to S with orthogonality of linear elements, have the following expressions : *

$$\begin{aligned} x_1 &= (U + V) \sin v - 2 \int \sin v \cdot V' dv, \\ (6) \quad y_1 &= -(U + V) \cos v + 2 \int \cos v \cdot V' dv, \\ z_1 &= -\frac{1}{a} [(U + V)u - 2 \int u \cdot U' du], \end{aligned}$$

where the accent denotes differentiation. From (6) we have that, when V is a constant, S_1 is a surface of revolution. Moreover, since these formulæ involve an arbitrary function of u , it follows that any surface of revolution can be defined by them.

Conversely, given a surface of revolution defined by

$$x = u \cos v, \quad y = u \sin v, \quad z = U;$$

the helicoid with plane director, whose equations are

$$\bar{x} = U_1 \sin v, \quad \bar{y} = -U_1 \cos v, \quad \bar{z} = av,$$

has the same axis and corresponds with orthogonality of linear elements, if

$$U_1 = au \int \frac{U'}{u^2} du,$$

where the accent denotes differentiation with respect to u .

By direct calculation we find from (6),

$$(7) \quad F_1 = \sum \frac{\partial x_1}{\partial u} \frac{\partial x_1}{\partial v} = \frac{V''}{a^2} [u(U + V) - (u^2 + a^2)U].$$

From (4) we see that the lines $u = \text{const.}$, $v = \text{const.}$ on S are asymptotic, and consequently the corresponding lines on S_1 form a conjugate system. Hence it follows from (7) that the necessary and sufficient condition that asymptotic lines on S cor-

* Bianchi, *Lezioni*, p. 276.

respond to lines of curvature on S_1 is that $V' = 0$, that is, S_1 must be a surface of revolution.

From (5) we see that in the latter case ϕ is a function of u alone. This, however, is a general property of the infinitesimal deformation of minimal surfaces. For, from the following formula, which we have established elsewhere,*

$$F_1 = F\phi^2 + \frac{1}{K^2(EG - F^2)} \left(D \frac{\partial \phi}{\partial v} - D' \frac{\partial \phi}{\partial u} \right) \left(D' \frac{\partial \phi}{\partial v} - D'' \frac{\partial \phi}{\partial u} \right),$$

it is seen that when S is a minimal surface referred to its asymptotic lines, the necessary and sufficient condition that the parametric lines on S_1 be the lines of curvature is that ϕ shall be a function of u alone or a function of v alone.

When in particular we take

$$(8) \quad U = \sqrt{u^2 + a^2}, \quad V = 0,$$

we get

$$x_1 = \sqrt{u^2 + a^2} \cdot \sin v, \quad y_1 = -\sqrt{u^2 + a^2} \cdot \cos v,$$

$$z_1 = -a \log(u + \sqrt{u^2 + a^2}),$$

which define the catenoid. From (5) we get $\phi = 1$, which is the case whenever, in the deformation of a minimal surface, the adjoint of the latter is taken for the surface S_1 .†

Genty‡ has shown that the cartesian coördinates, x_0, y_0, z_0 , of the associate surface § S_0 in an infinitesimal deformation are given by the equations

$$dx_1 = z_0 dy - y_0 dz,$$

$$dy_1 = x_0 dz - z_0 dx,$$

$$dz_1 = y_0 dx - x_0 dy.$$

Substituting the expressions for $x, y, \dots, z_1$, from (1) and (6), and solving we find

$$(9) \quad \begin{aligned} x_0 &= \frac{1}{a} [(U + V - uU') \sin v + V' \cdot \cos v], \\ y_0 &= \frac{1}{a} [-(U + V - uU') \cos v + V' \cdot \sin v], \\ z_0 &= U'. \end{aligned}$$

* *Amer. Jour. of Math.*, vol. 24, p. 177.

† *Ibid.*, p. 192.

‡ *Toulouse Annales*, vol. 9.

§ Bianchi, *l. c.*, p. 279.

The linear element of S_0 is readily found to be

$$(10) \quad ds_0^2 = \frac{U''^2}{a^2} (u^2 + a^2) du^2 + \frac{1}{a^2} (V'' + V - uU' + U)^2 dv^2.$$

It is well known that the lines upon any associate surface corresponding to the asymptotic lines on S form a conjugate system. From (10) we see that the conjugate system on S_0 corresponding to asymptotic lines on S are the lines of curvature. Furthermore, the lines of curvature $v = \text{const.}$ are geodesics and consequently S_0 is a surface of Monge.* From the form of the coefficient of dv^2 in (10) we have that the generating developable is a cylinder.† Hence

In any infinitesimal deformation of a skew helicoid the associate surface is a moulure surface.

Conversely, given any moulure surface; its equations can be put in the form (9) and then can be taken for the associate surface in the deformation of the helicoid (1), corresponding to the value $(U + V)(u^2 + a^2)^{-\frac{1}{2}}$ of the characteristic function.

From (6) and (9) we get the

THEOREM: *When the surface S_1 in an infinitesimal deformation of a skew helicoid is a surface of revolution, the associate surface S_0 also is a surface of revolution, and their lines of curvature correspond.*

And conversely,

When the associate surface S_0 is a surface of revolution, the characteristic surface S_1 is a surface of revolution.

When in particular S_1 is the catenoid, S_0 is the sphere of radius unity and center at the origin.

If we put

$$U = a \sqrt{u^2 + a^2} - au \log(u + \sqrt{u^2 + a^2}), \quad V = 0,$$

the formulæ (9) define the catenoid. We have shown‡ that the necessary and sufficient condition that the lines of curvature be unaltered in the deformation of S is that S_0 be the adjoint minimal surface of S . Hence, when

$$\phi = a \left[1 - \frac{u}{\sqrt{u^2 + a^2}} \log(u + \sqrt{u^2 + a^2}) \right],$$

* Monge, *Applic. de l'Analyse a la Géométrie* 5 ed., chap. 25.

† Darboux, *Leçons*, vol. 1, p. 105.

‡ L. c., p. 199.

$$(2) \quad \begin{array}{ccc|ccc} a_{11} & \cdots & a_{1k} & 0 & \cdots & 0 \\ . & . & . & . & . & . \\ a_{k1} & \cdots & a_{kk} & 0 & \cdots & 0 \\ \hline a_{k+1,1} & \cdots & a_{k+1,k} & a_{k+1,k+1} & \cdots & a_{k+1,n} \\ . & . & . & . & . & . \\ a_{n1} & \cdots & a_{nk} & a_{n,k+1} & \cdots & a_{nn} \end{array}$$

When a linear differential equation

$$(3) \quad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0$$

is given, we fix a domain of rationality and now if equation (3) reduces within this domain to

$$(4) \quad \frac{d^k y}{dx^k} + q_1 \frac{d^{k-1} y}{dx^{k-1}} + \cdots + q_k y = 0 \quad (k < n)$$

the group (2) will be isomorphic with the group of rationality of the equation (3).

If $k = n - 1$, then by effecting a quadrature equation (3) will reduce to

$$(5) \quad \frac{d^{n-1} y}{dx^{n-1}} + r_1 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + r_{n-1} y = 0$$

and the group (2) will become

$$(6) \quad \begin{array}{cccc|ccc} a_{11} & \cdots & a_{1,n-1} & 0 & & & \\ . & . & . & . & . & . & \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & 0 & & & \\ a_{n,1} & \cdots & a_{n,n-1} & a_{nn} & & & \end{array}$$

If now (5) reduces within the same domain of rationality to

$$(7) \quad \frac{d^{n-2} y}{dx^{n-2}} + s_1 \frac{d^{n-3} y}{dx^{n-3}} + \cdots + s_{n-2} y = 0$$

the group (6) becomes

$$\begin{array}{ccccccc}
 & a_{11} & \cdots & a_{1, n-2} & 0 & 0 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 (8) & a_{n-2, 1} & \cdots & a_{n-2, n-2} & 0 & 0 \\
 & a_{n-1, 1} & \cdots & a_{n-1, n-2} & a_{n-1, n-1} & 0 \\
 & a_{n, 1} & \cdots & a_{n, n-2} & a_{n, n-1} & a_{nn}.
 \end{array}$$

When equation (3) is integrable by quadratures this process can be continued n times; the group we are considering (which is isomorphic with the group of rationality) takes the form

$$\begin{array}{ccccccc}
 & a_{11} & & & & & \\
 & a_{21} & a_{22} & & & & \\
 (9) & a_{31} & a_{32} & a_{33} & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & a_{n-1, 1} & a_{n-1, 2} & \cdots & a_{n-1, n-1} & & \\
 & a_{n, 1} & a_{n, 2} & \cdots & a_{n, n-1} & a_{n, n}.
 \end{array}$$

But (Lie-Engel, Transformationsgruppen I, chapter 27) the group (9) is an integrable group.

The Jordan-Beke condition for reducibility, employed above, is both necessary and sufficient. We have thus deduced as a special case of the Jordan-Beke theorem the theorem of Vessiot, namely, "*the necessary and sufficient condition that a linear homogeneous differential equation shall be integrable by quadratures is that its group of rationality be integrable.*"

PHILADELPHIA, PA.

August, 1902.

THE CENTENARY OF THE BIRTH OF ABEL.

AT the close of the first week in September last, the Royal Frederick University at Christiania, Norway, celebrated the one hundredth anniversary of the birth of Niels Henrik Abel. The occasion was noteworthy as the first international academic celebration in Norway and was in every way instructive and enjoyable for the delegates and guests of the university. On their arrival at the station they were met by representatives of the university who had been previously instructed as to which language each of them spoke and who conducted them to the rooms to which they had been assigned in the various hotels,

acting the while as interpreters or informants and talking with apparent indifference French, German, English, or Norwegian. Each delegate was presented with two large volumes of which one was : Norway, the official publication of the country at the time of the Exposition at Paris in 1900, and the other : Niels Henrik Abel, *Mémorial publié à l'occasion du centenaire de sa naissance*, containing an ode to Abel by Bjoernson, a historical account of his life and correspondence by Holst, the letters written or received by Abel translated into French with notes and with facsimile reproductions, letters and documents relative to him likewise with notes, the original text of such letters as Abel wrote in Norwegian, and an essay by Sylow on Abel's studies and discoveries with further facsimiles. It is this volume which for most of us will probably be the most cherished and permanent monument of the celebration. There are three other volumes now publishing which may be mentioned here as memorials of the occasion and which will doubtless be of a greater value from a purely scientific point of view than those presented to us by the university. These are volumes XXVI, XXVI *bis*, XXVI *ter*, of the *Acta Mathematica*, which will contain recently discovered and hitherto unpublished memoirs by Abel and original memoirs by a great number of mathematicians upon those branches of mathematics in which Abel was especially interested.

Although there was an informal reception for the delegates on Thursday evening, September 4, in St. Hanshaugen Park with an introductory address of welcome by Professor Nansen, chairman of the Abel committee, the official opening of the celebration took place Friday at noon under the presidency of the rector of the university and the president of the academy of sciences. The programme consisted of a beautiful cantata composed especially for the occasion by Bjoernson and Sinding, of addresses by the minister of state and the rector of the university with responses by Professors Heinrich Weber and Volterra on behalf of the delegates, and of an address or essay by Professor Sylow on the work of Abel. Professor Sylow alone spoke in Norwegian and copies of his address translated into French were distributed among the delegates.

In the evening his majesty, King Oscar II, who in person attended the official meetings of the celebration, tendered a supper to the delegates and other invited guests. The long informal receptions before and after the supper afforded the first

and probably the best opportunity which we had to become acquainted with one another.

On Saturday at noon the second official meeting was held under the auspices of the university. This was more of an academic, less of a state affair, than that on the preceding day. The president of the academy of sciences delivered an address. Forsyth, Gravé, Picard, Schwarz, Zeuthen responded on behalf of the delegates, Hensel for *Crelle's Journal*, Mittag-Leffler for the *Acta Mathematica*. The delegates then filed by and presented the inscribed congratulatory addresses or the words of felicitation sent by the various societies or universities represented.

It is difficult to obtain an exact list of the delegates, for some, Peano, Pincherle and Padé, for example, who were expected and who would have added to the already brilliant assemblage, did not arrive; and others, as Selivanov and Knott, that one could notice came unofficially or too late to be included in the lists. The correspondents of the press give the following list taken down as the delegates filed by.

From Germany and Austria: Schwarz, H. Weber, Pringsheim, Hensel, Hilbert, Seeck, Kiepert, Fr. Meyer, Krause, Engel, Gutzmer, Lerch, Czuber. From Belgium: Mansion, Paige. From Holland: Cardinaal, Kristensen. From Denmark: Thiele, Zeuthen. From France: Picard and E. Sauvage. From America: Canfield, Olsen, Spangler, Watson (not mathematicians representing various universities), and Fields, Newcomb, Van Vleck, Wilson (the last two representing the AMERICAN MATHEMATICAL SOCIETY). From Switzerland: Graf, Fehr. From Greece: Stephanos. From Russia and Finland: Lindelöf (senior), Donner, Tallqvist, Bäcklund, Zdanov, Gravé, Zantshevsky. From Sweden and Norway: A. B. Bäcklund, Wijkander, Möbius, Théel, Bendixson, von Koch, Dunér, Mittag-Leffler, Dahle, Hansen, Lugaard, Spaare. From Italy: Orland, Volterra, Del Pezzo. From Great Britain: Lamb, Love, Forsyth, Joly, Sampson, Greenhill, Hobson, Jack.

The rector of the university then delivered an address preliminary to conferring the honorary degrees. By the original charter of the university the power of bestowing such degrees had been withheld. For this celebration, however, special permission had been granted and to signalize the occasion further a new degree, Doctor Mathematicæ, was introduced. This distinctive degree was bestowed upon the following twenty-nine

persons of whom the ten who were present went forward as their names were read and received their diplomas amid great applause :

Appell, Bäcklund, Boltzmann, G. Cantor, Cremona, Darboux, G. Darwin, Dedekind, Dini, Forsyth, J. W. Gibbs, Hilbert, Jordan, Lord Kelvin, Klein, Koenigsberger, Markov, Mittag-Leffler, Newcomb, Picard, Poincaré, Lord Rayleigh, Salmon, Schwarz, Stokes, Volterra, Weber, Wirtinger, Zeuthen.

Professor Newcomb made a short speech thanking the university on behalf of the recipients of the degree. The rector of the university delivered a peroration and declared the celebration officially at an end.

But for the delegates much was yet in store — perhaps the pleasantest part of all. Professor Bjerknes and his son took some of us who were especially interested in mechanics, among whom could be seen Volterra, Love, and Lamb, to his laboratory and showed us his apparatus specially devised for exhibiting his hydrodynamic action at a distance. In the evening the city of Christiania gave us a dinner, during and after which we had another excellent opportunity to renew or further our acquaintance with one another. This occasion was the climax of the celebration. A magnificent collation was served. The King and the various nationalities were toasted. Enthusiasm waxed to fever heat. Finally the students came in a torch-light procession under the windows of the banquet hall, greeting the delegates with prolonged and uproarious applause. On Sunday various excursion parties into the environs of Christiania were formed for our interest. Sunday evening there was a special representation of Ibsen's *Peer Gynt* at the national theater to which we had invitations. For the following Wednesday Professor Mittag-Leffler extended an invitation to lunch to such of us as might be in Stockholm.

By Monday evening most of us had left Christiania feeling that the celebration had been uncommonly successful : for the committee had accomplished the difficult task of making each succeeding event better than the previous one.

E. B. WILSON.

PARIS, September 25.

THE ENGLISH AND FRENCH TRANSLATIONS OF HILBERT'S GRUNDLAGEN DER GEOMETRIE.

1. *Les Principes fondamentaux de la Géométrie.* Par D. HILBERT. Traduit par L. LAUGEL. Paris, Gauthier-Villars, 1900. 4to. 111 pp.
2. *The Foundations of Geometry.* By DAVID HILBERT. Translated by E. J. TOWNSEND. Chicago, The Open Court Publishing Company, 1902. 8vo. 132 pp.

IT is indeed a matter for congratulation that Professor Hilbert's masterly discussion of the foundations of geometry has become so well known and so widely circulated. This circumstance is undoubtedly due to its clearness and force; for, while some of the previous studies along similar lines are difficult even for the advanced student to understand, Hilbert's style is so deceivingly clear as to lead certain minds to predict the use of this book in elementary instruction.

An excellent review of the original,* rendered into English by Professor Ziwet, was given in the BULLETIN (volume 6, 1900, pages 287-299) by Dr. Sommer, of Göttingen; the present review will therefore not deal with the German edition, except for purposes of comparison.

If a minute criticism of the language of the French translation were the main purpose, the reviewer would certainly feel great hesitation in undertaking the task. But there are certain additions to the French translation which are most noteworthy, and with these we shall occupy ourselves chiefly.

On page 291 of his review, mentioned above, Dr. Sommer explains that Hilbert's axiom V is not sufficient to furnish a satisfactory foundation for the complete discussion of "the continuity of the straight line in the ordinary sense." This lack is supplied in the French edition by an additional axiom (by Hilbert) entitled "Axiome d'intégrité" (Vollständigkeitsaxiom), which practically requires that the system already set up shall

* In this connection, a review of Hilbert's *Grundlagen der Geometrie* by H. Poincaré, *Bull. des Sciences Math.* (2), vol. 26 (Sept., 1902), p. 249, should be mentioned. This review is, as would be expected, of the greatest importance. The present writer regrets that his review was in type before that of Poincaré was seen. Footnotes have been added occasionally at points where this review would have been most influenced by Poincaré's.

be considered as closed, *i. e.*, incapable of extension by the adjunction of additional elements. This axiom corresponds to the one introduced by Hilbert (*Jahresbericht der Deutschen Mathematiker Vereinigung*, 1900) for arithmetic; and is certainly radically different, in the character of the assumption made, from the other axioms proposed by Hilbert. Its introduction enables us to prove the fundamental theorem of Bolzano, but the reviewer feels that its form is scarcely final, and that it is not wholly satisfactory to Hilbert himself.*

Another addition to the French edition merits more attention than it seems to have received; the original conclusion of the German edition is replaced by a new appendix (pages 106–111), written by Hilbert, which shows clearly the importance of his work for non-euclidean geometry in its broadest sense, a point upon which Hilbert dwelt to a considerable extent in his lectures preceding the publication of the original *Festschrift*, but which is scarcely mentioned in the *Festschrift* itself. The apparent reason for the tendency of non-euclidean geometers in the past to cast out the axiom of parallels alone, is that this was thought to be the only axiom *explicitly* assumed by Euclid which could not be directly tested by physical experiment.† Another axiom, or set of axioms, namely those regarding continuity, assumed *implicitly* by Euclid, is, however, equally open to philosophical exception.‡ The effect of the omission of the axiom of Archimedes (Hilbert, V) was therefore of peculiar interest to Hilbert, and the subject was discussed considerably in his lectures. A student, Dr. Dehn, completed these inquiries, and a review (by Hilbert) of Dr. Dehn's thesis forms the major portion of this addition to the French translation. The one striking fact that *a geometry is possible in which the sum of the angles of any triangle is two right angles and in which similar non-congruent triangles exist, while an infinity of parallels to any straight line may be drawn through any point*, will sufficiently indicate the remarkable nature of Dr. Dehn's results.

This geometry is called "semi-euclidean," and is evidently non-archimedean, the axioms III and V being omitted; but in the whole discussion the axioms I, IV and II (in modified form) are assumed to hold. To the reviewer it would seem, from the theorem just mentioned, that the assumption that the

* See also Poincaré, l. c., pp. 256, 271, 272.

† The reviewer disclaims any intention of saying that this is a logically valid reason; it is merely the *actual* reason.

‡ See Poincaré, l. c., pp. 250, 258.

sum of the angles of a triangle is two right angles would be more advisable than the form of the parallel axiom given by Euclid (Hilbert III), since it is demonstrated that it is a "weaker" axiom, in the sense of Professor Moore. For while it can be proved from the axiom III, without the use of axiom V, we must include V in order to prove III from it. At any rate, as Dr. Dehn remarks, this substitute for the axiom of parallels is conclusively shown not to be equivalent to Euclid's form; and, while Euclid probably deserves no praise in the matter, his was certainly a happy choice in that he possessed Archimedes's axiom only by implicit assumption. While this subject merits considerable discussion, we cannot enter into further details without passing from the book in hand. Still, Dr. Dehn's thesis (*Mathematische Annalen*, volume 53 (1900)), as also his subsequent papers in the *Annalen*, form a direct and important addition to Hilbert's memoir.

Aside from these additions, the French edition is simply a remarkably good translation. The translator has happily given himself just enough liberty to render the spirit of the German into idiomatic French; but the reviewer feels confident that the sense has always been perfectly preserved. It is especially noteworthy that M. Laugel had the direct and enthusiastic assistance of Hilbert himself in the preparation of the French edition.

In spite of the hesitation expressed above, the reviewer feels moved to enquire whether better translations for "Axiome der Verknüpfung" (axiomes d'association), and for "Axiome der Anordnung" (axiomes de distribution), could not have been found, especially in view of the necessity of using these same terms in describing the corresponding sets of axioms of arithmetic in Chapter III, where the associative, distributive, and commutative laws of arithmetic are all placed under the head of "associative" axioms, whereas the "distributive" axioms are of an entirely different nature.* Finally the translations of "flächengleich" (égaux par addition), and of "inhaltsgleich" (égaux par soustraction), are certainly widely divergent from the original, and might possibly have been improved upon; but it may be contended that they are justifiable as a closer description of the ideas represented.

Of the typography nothing need be said, further than that the book is printed by the firm of Gauthier-Villars.

* Poincaré, l. c., uses "axiomes projectifs" and "axiomes de l'ordre." He expressly discards "axiomes de connection" as a translation of "Axiome der Verknüpfung."

On the whole, on account of the additions mentioned above, and in view of the faithful reproduction of the spirit of the original, the French edition compares very favorably with the German, for those who understand both languages equally well. The reviewer had excellent success in the use of the French edition by a portion of his class last winter, the students having been advised to procure either the French or the German editions at their discretion.

The English translation is the work of one who had the advantage of hearing Hilbert's preparatory lectures. It contains the additions to the French translation, but there are no new additions. Some unimportant omissions are made, including the original conclusion (also omitted in the French edition). In a short preface Professor Townsend expresses his own views upon the important features of the work. It is unfortunate that he does not emphasize the radical departures of Hilbert's *method* of presentation — especially the power obtained by considering the axioms as fundamental, the conceptions of point, line, and plane as trivial.* Notice is taken in footnotes of some recent memoirs commenting on Hilbert's work.

Some features of the translation are particularly happy. Especially does the rendering of "Widerspruchslosigkeit" by "compatibility" appeal to the reviewer. The translations of "flächengleich" (of equal area), and of "inhaltsgleich" (of equal content), §18, etc., are certainly nearer the original than the expressions used by the French translator. Where custom has established a translation of a standard technical word, Professor Townsend generally appears to have used it, as, for example, "set" for "Menge," "enumerable" for "abzählbar," "(number) field" for "(Zahl) körper," to all of which excellent alternate expressions are, however, in common use. The expression "numerical multiplicity," translated on page 126 from the French "multiplicité numérique," suffers on the other hand by comparison with Ziwet's† "manifold of numbers" as an equivalent for the German word "Zahlenmannigfaltigkeit."

* To illustrate: Hilbert, in his lectures, used effectively a geometry in which "points" are the ordinary positive integers, and "lines" are the ordinary negative rational numbers. The principle involved was, of course, used before Hilbert by others, but it has taken a far more tangible form in Hilbert's hands. See Poincaré, l. c., pp. 250, 252, etc.

† In his translation of Sommer's review above mentioned.

Professor Halsted * informs us that the translation of "Anordnung" *must* be "arrangement," but the reviewer feels that several other words are also suitable translations of "Anordnung," in its technical sense, just as "ausgezeichnet," when used technically, need not necessarily be rendered "remarkable."† The translations "order," "arrangement," and "order of sequence" are to be found in Professor Townsend's edition, and any one of them, used consistently, would have been acceptable. Professor Ziwet, in the review mentioned above, uses "order." Perhaps even a still better word might be found, but Professor Townsend is to be congratulated in not having followed the French translator in his unhappy choice. The omission of the heading "Erklärung" can be defended, although the reviewer personally inclines to its retention.

The work is unfortunately somewhat marred by several errors, some of which, occurring at critical points, are apt to be misleading. On page 24, line 11, the word "all" is wholly wrong, as the translator might have known from Hilbert's lectures or at least from Dr. Sommer's review. It may be admitted that the original is not entirely clear on this point, but no such word as "all" occurs in the German edition. The translation of "einzig" by "definite" in line 8, page 53, may possibly be confusing; *any* single point is to determine a segment 0. The word "only" must be inserted between "which" and "such" on line 11 from the bottom of page 98. On page 107, line 1, we are not to understand that Pascal's theorem is *identical* with the commutative law, but merely that the truth of the commutative law insures that Pascal's theorem holds, under the given conditions.‡ On page 124, line 11, we understand that the extraction of the square root of f_3 is the third in order, *not* that it is now to be performed "a third time." The statement on page 125, that "every regular polygon can be

* In a brief review of Professor Townsend's edition, *Science*, August 22, 1902. The review is not especially appreciative and dwells on several mistakes and misprints. Some of the rather harsh criticisms are quite unfounded, as, for instance, those regarding axiom IV, 1, p. 12; theorem 16, p. 22; and a sentence on page 25, lines 26-28; while others seem trivial, for example, that of "emanating," p. 13, etc. The errors on pp. 24 and 125, noticed by Professor Halsted, and the omission of the heading "Erklärung," are considered later in the present review. The remainder of Professor Halsted's criticisms seem unimportant, in the sense that the passages in question are at least never misleading as they stand, and certainly do not call for separate reconsideration here.

† An accepted translation for "ausgezeichnete Untergruppe" is "self-conjugate subgroup."

‡ See Poincaré, l. c., p. 267.

constructed by the drawing of straight lines and the laying off of segments" will scarcely mislead any one. It is conceivable that this curious mistranslation was caused by some oversight, but it seems that "jene" has been mistaken for "jede(s)": "Every regular polygon" should, of course, be replaced by "the above-mentioned regular polygons."

But the most dangerous errors occur in the translation of the important new appendix of the French edition. On page 127, lines 17, 19, 20, 21, "par" is translated "from" instead of "by," and on page 130, line 11, "vis-a-vis de" is also rendered "from" instead of "with respect to," while on page 128, line 9, "imposée" is changed into "confined." A reading of the paragraphs in question will disclose the fact that at least the first two of these mistakes totally destroy all meaning at critical points. These errors, and some minor ones, make this conclusion the poorest part of the English edition.

The translator's English sometimes lacks the true idiomatic ring, the worst instances being, perhaps, the footnote on page 85, and the sentence which begins on line 7, page 124. There are also several minor faults of translation, which will fortunately not mislead the student. Among these are changes from singular to plural nouns and *vice versa*, for instance, "square root(s)," line 23, page 121; the insertion and omission of words, as "consequently" line 21, page 127; and mistranslations of words. Of such minor errors the reviewer has noticed, not counting duplicates, nineteen.

Among the misprints found only two are serious: the reference to "§ 30" on page 98, line 24, should be "§ 27"; and "and" three lines from the bottom of page 123, should be "any." The misprint (or error) in the original in formula (1), § 33, corrected in the Errata (page 111) of the French edition, is corrected properly on page 104 of Professor Townsend's translation.

The word "all" on page 74, line 22, while not strictly a mistranslation of language, gives rise to an impression which is faulty, and which is not intended in the original. For it is nowhere proved that the assumption of each of the axioms IV, 1-5, is necessary for the proof of Desargues's theorem. The proper impression would possibly be conveyed by substituting the phrase "or the entire set of axioms of congruence" for the corresponding phrase in the text; the idea being that, whereas we have assumed axioms IV, 1-5, it is *necessary* to *add* to

these axiom IV, 6, but it is not shown that we need *retain* each of the axioms IV, 1-5.

The notation for a segment joining two points, *e. g.*, 0 and $\frac{1}{2}$, has been changed in § 17, perhaps for the better, from $0\frac{1}{2}$ to $(0, \frac{1}{2})$. There are also other slight changes of notation, which are generally commendable. But a great deal is lost by not following Hilbert's italicization of words and of some of the theorems, which enabled him to emphasize what he considered most important. In particular the words "NON PAS" of the principal theorem of the new conclusion to the French edition, which Hilbert and Laugel emphasized not only by extraordinary type, but also by extraordinary position in the sentence, are rendered on the last line of page 128, by an ordinary "not," in ordinary type.

The Open Court Publishing Company deserves praise for continuing to publish translations of foreign scientific classics into English. The book is well printed, except for a few bad figures, of which the worst are figures 18 (page 41) and 38 (page 76), the latter being actually misleading. The type is good, excepting the identity sign. With regret we are compelled to notice, however, that the edition is not wholly satisfactory on account of the errors mentioned above. In its present form it can scarcely be recommended to students unless they can read neither German nor French; and then it should be used only under competent guidance. But most of the serious errors could be corrected by changing a few words, and if a page of such corrections were inserted in the volume it could be used by students with greater safety.

To other than university graduate students *no* edition of the book appeals. For the statement* that it has pedagogical value, or that it is to be of influence on elementary instruction, only means that students of very advanced mathematics, by their knowledge of it, may reflect into elementary books or teaching which they may influence, something of its general spirit, almost nothing of its actual contents. To insert the system of axioms proposed by Hilbert in an elementary (high school) book or, for instance, to try to introduce into such a text-book the theory of motion as based upon congruence, or even the proof that all right angles are equal, would be as ill considered as to expect the average high school teacher to grasp the meaning of the original in its entirety.

*Sommer, l. c., p. 299; Townsend, preface to English edition.

We must rejoice, however, in proof of the wide circulation of Hilbert's ideas, that both a French and an English translation have actually been published. A widely diffused knowledge of the principles involved will do much for the logical treatment of all science and for clear thinking and clear writing in general.

E. R. HEDRICK.

YALE UNIVERSITY,
September, 1902.

DICKSON'S LINEAR GROUPS.

Linear Groups with an Exposition of the Galois Field Theory.

By L. E. DICKSON, Assistant Professor of Mathematics in the University of Chicago. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Volume VI. Leipzig, B. G. Teubner, 1901. 8vo, x + 312 pp.

SHORTLY after the appearance of the first few numbers of the *Encyclopädie der Mathematischen Wissenschaften* the publishers announced a series of text-books on advanced mathematics to be issued in connection with the *Encyclopädie*. While the authors of articles in the *Encyclopädie* were especially requested to take advantage of this series to develop their subjects more fully and thus make them more accessible to the student, other writers were asked to assist to make the series as complete as possible. More than fifty different volumes of this series have already been announced, by almost as many different writers of various countries.

Never before has there been such extensive collaboration to bring the developments in the various parts of our subject within the reach of the student. It is hoped that this series will do much towards increasing the number of well-equipped investigators and thus exert a strong influence towards more substantial progress in various directions. The fact that the authors belong to so many different countries emphasizes the cosmopolitan element in mathematical work and the absence of national prejudices among its devotees.

The present work is the sixth volume of the series and is devoted to a subject which has been developed principally on French and American soil. The fundamental ideas are due to Galois and were published by him at the early age of eighteen

in a memoir entitled "Sur la théorie des nombres."* In this brief memoir Galois introduced a new kind of imaginaries and studied their classification and reduction to the least possible number.

Let $F(x)$ represent an integral function with integral coefficients. If it is possible to find three other integral functions with integral coefficients $F_1(x)$, $F_2(x)$, $F_3(x)$ such that the degrees of $F_1(x)$ and $F_2(x)$ are less than the degree of $F(x)$ and that

$$F(x) = F_1(x)F_2(x) + pF_3(x)$$

then $F(x)$ is said to be reducible modulo p . Otherwise it is said to be irreducible. In the latter case Galois represents a root of the congruence $F(x) \equiv 0 \pmod{p}$ by i and considers the general expression

$$a + a_1i + a_2i^2 + \cdots + a_{n-1}i^{n-1},$$

where $a, a_1, a_2, \dots, a_{n-1}$ represent integers and n is the degree of $F(x)$. He observes that the $p^n - 1$ values of this general expression, when the coefficients assume separately all the least positive residues modulo p such that not all are simultaneously equal to zero, are powers of a single one of them. Hence all the algebraic quantities which enter into the theory of these imaginaries are roots of the equations of the form $x^{p^n} = x$ and are therefore independent of the form of the special irreducible congruence of degree n .†

From these results it is evident that the p^n values of the general expression of the preceding paragraph constitute a finite domain of rationality, endlicher Körper, or finite field. The interest in this theory has been greatly enhanced by Moore's proof of the fact that the elements of every finite field may be put in a (1, 1) correspondence with those of such a Galois field.‡ The great simplicity of this field appears perhaps more remarkable when compared with the difficulties which are met in the study of other finite systems, such as Kronecker's modular systems and the theory of groups of finite order.

The first part (pages 3–71) of the present work is devoted to an exposition of the Galois field theory, chiefly in its abstract

* *Bulletin des Sciences de M. Férussac*, vol. 13 (1830), p. 428. Reprinted in *Liouville* vol. 11 (1846), p. 398.

† *Oeuvres de Galois, Liouville*, vol. 11 (1846), p. 400.

‡ This term seems to have been first used by Moore, *BULLETIN*, vol 3 (1893), p. 73.

form. Beginning with the simplest example, classes of residues with respect to a prime modulus, the author points out the characteristic properties of a Galois field. The greater part of the first chapter is devoted to the proof that no other finite field can exist and that there is only one such field of a given order p^n . The notation and general method of proof are practically the same as those employed by Moore in his paper entitled "A doubly infinite system of simple groups." *

The second chapter is devoted to a proof of the existence of a Galois field of order p^m , $GF[p^m]$, for every prime p and every integer m . This existence follows directly from the fact that there is at least one congruence of degree m which is irreducible modulo p . Instead of proving the latter fact directly as Serret has done,† the author assumes the existence of a $GF[p^n]$ and proves that there are irreducible functions of every degree in this field. Since the field of integers modulo p is known to exist, this not only proves the point in question but also exhibits some important properties of the $GF[p^m]$ with respect to the included $GF[p^n]$. A somewhat different method of proof is indicated in the "exercises" which close the chapter.

The determination and classification of irreducible quantics forms the subject matter of the third chapter. Just as in the study of least residues with respect to a prime modulus, it is convenient to make prominent use of the law of exponents in the study of the irreducible quantics. A quantic $F[x]$ of degree m and irreducible in the field of order p^n is denoted by $IQ[m, p^n]$. It is said to belong to the exponent e provided e is the least positive integer for which $F(x)$ divides $x^e - 1$ in the $GF[p^n]$. When $e = p^{nm} - 1$ then $F(x)$ is said to be a primitive irreducible quantic of degree m in $GF[p^n]$. The determination of such quantics forms one of the most difficult problems in the Galois field theory. The latter part of the chapter is devoted to special methods of solving this problem.

Chapter IV is devoted to miscellaneous properties of Galois fields and begins with the study of squares, not-squares, and m th powers in a Galois field. To some readers this part would doubtless have been clearer if they were told explicitly that they were concerned with operators of a cyclic group C . The statement that "every mark has one and only one square root in the $GF[2^n]$ " would thus appear as a special case of the theorem

* Moore, Published Papers of the Am. Math. Soc., vol. 1 (1896), p. 208.

† Serret, *Algèbre supérieure*, vol. 2 (1885), p. 137.

that in an abelian group of odd order every operator is the square of one and only one operator. It is true the zero element of $GF[2^n]$ is not contained in C but the theorem evidently applies to it. Moreover, it may be observed that 0 and 1 are the only two marks of $GF[2^n]$ which coincide with their square roots.

Similarly, Section 62 is equivalent to saying that all the operators of a subgroup of even index are squares of operators of C . The theorem of Section 63 is clearly true for any abelian group and is a direct consequence of the facts that in an abelian group there is a subgroup whose order is any arbitrary divisor $[(p^n - 1)/d]$ of the order of the group and if each operator of an abelian group is raised to a power prime to the order of the group the resulting operators may be put in a (1, 1) correspondence with the original operators. In other words, every operator of an abelian group is the m th power of just one of its operators, where m is any number prime to the order of the group.

In such developments it is always a question what an author should presuppose. If a knowledge of the fundamental abstract properties of abelian groups had been assumed on the part of the reader it would have been possible at various places to indicate more clearly the contact with abstract group theory and also to bring out additional facts in the same amount of space notwithstanding the fact that the zero element in the Galois field requires separate treatment from this standpoint.

Sections 64-67 are devoted to generalizations of Jordan's results in regard to the solutions of certain quadratic equations in a Galois field. This is followed by a study of the additive groups and their multiplier Galois fields as employed by Moore. A set of m marks $\lambda_1, \lambda_2, \dots, \lambda_m$ belonging to the $GF[p^n]$ and linearly independent with respect to the $GF[p]$ give rise to p^m distinct marks

$$c_1\lambda_1 + c_2\lambda_2 + \dots + c_m\lambda_m \quad (c_i = 0, 1, \dots, p - 1)$$

with respect to the larger field. These are said to form an additive group of rank m with respect to the $GF[p]$. In particular, the $GF[p^n]$ may be regarded as an additive group of rank n . These conceptions are generalized and a condition for the linear independence of marks with respect to an included field is developed. The chapter closes with a consideration of

Newton's identities for sums of powers of the roots of an equation belonging to a Galois field.

In this chapter the term *group* is used for the first time but no definition is given. The statement "since the *sum* of any two of these p^m marks may be expressed as one of the set, they are said to form an additive group" cannot be regarded as a definition. Moreover, it is apt to mislead the reader since such incomplete definitions of group are not uncommon in the mathematical literature. On the other hand, if the reader is supposed to be familiar with the definition of the term it would have been of interest to contrast this additive group with the multiplicative group formed by the marks of a field, which differ from zero.

The fifth and last chapter of Part I is devoted to the analytic representation of substitutions on marks of a Galois field. This subject is treated quite fully in Part I of the author's dissertation * where extensive references are given. Beginning with the definition of a substitution quantic the author develops the analytic conditions which characterize such functions and gives a table of all such reduced quantics whose degree is less than 6. A study of the Betti-Mathieu group and six suggestive exercises furnish the close of these interesting and important developments of modern algebra.

The second part of the present volume (pages 75-310) is devoted to a study of the most important properties of linear groups in a Galois field. Some of these groups were investigated in the field of integers modulo p by Galois, Serret and Jordan. These results are here generalized for the larger field and new systems are investigated ab initio in this field. The work of Moore first emphasized the importance of employing the general Galois field in linear group problems as the investigations are generally not much more complicated and the results are more general. Jordan frequently indicated these generalizations without entering upon their complete development.

A great part of the text is taken directly from the author's numerous memoirs, but in other parts the method of presentation differs widely from that employed in the original papers. More stress seems to have been laid upon clearness in presentation and some of the longer calculations have been avoided. The work has thus gained considerable in attractiveness and

* Dickson, *Annals of Mathematics*, vol. 11 (1897), p. 65.

has become more suitable to serve its purpose as an introduction to the extensive subject of linear groups in a finite field.

The first chapter begins with two definitions of the general linear homogeneous group on m indices with coefficients in the $GF[p^n]$. The group is denoted by $GLH(m, p^n)$ while the symbol $GLH[m, p^n]$ is used to represent its order. The latter is proved to be

$$(p^{nm} - 1)(p^{nm} - p^n)(p^{nm} - p^{2n}) \cdots (p^{nm} - p^{n(m-1)}).$$

For $n = 1$ the factors of composition were determined by Jordan in his *Traité des Substitutions*. For the general value of n they were determined independently by the author and by Burnside. The general case is considered here, and from these factors it follows directly that the group of all linear fractional substitutions $LF(m, p^n)$ in the $GF[p^n]$ on $m - 1$ variables and having the determinant either unity or some m th power in the field, is simple except in the two special cases $(m, p^n) = (2, 2)$ or $(2, 3)$.

The abelian linear group and a generalization of this group form the subjects of Chapters II and III. The former has been investigated by Jordan in the field of integers, while the latter is mentioned by him but he did not investigate its properties. Both have been studied by Dickson in the general Galois field and are here presented in this general way. The term abelian was first applied to these groups by Jordan, although they are not, in general, commutative. To distinguish them from commutative groups they are called abelian *linear* groups.

The hyperabelian group consists of the totality of linear homogeneous substitutions in the $GF[p^{2n}]$ which leave absolutely invariant the function

$$\psi = \sum_{i=1}^n \begin{vmatrix} \xi_{2i-1} & \xi_{2i} \\ \xi_{2i-1}^{p^n} & \xi_{2i}^{p^n} \end{vmatrix}.$$

It was first studied by the author in the *Proceedings of the London Mathematical Society*, volume 31, and forms the subject of Chapter IV of the present volume. It is distinct from the hypoabelian groups studied by Jordan in his *Traité* and by Dickson in the *BULLETIN* and elsewhere. Moreover, it must be distinguished from Picard's hyperabelian group of infinite order. The totality of its substitutions whose coefficients belong to the included $GF[p^n]$ constitute the special abelian

linear group $SA(2m, p^n)$. The latter is transformed into itself by a subgroup of the hyperabelian group whose order is $p^n + 1$ times the order of $SA(2m, p^n)$.

The greater part of Chapter V is taken directly from the author's article entitled "The structure of the linear homogeneous groups defined by the invariant $\lambda_1 \xi_1^r + \lambda_2 \xi_2^r + \dots + \lambda_m \xi_m^r$."* Chapter VI is devoted to a new exposition of the theory of compounds of a linear homogeneous group. This theory is entirely due to the author, having been introduced by him in several papers published in 1898.† He also employed it in his new definition of the general abelian linear group, which was published in the first volume of the *Transactions of the AMERICAN MATHEMATICAL SOCIETY*.

In Chapters VII and VIII the linear homogeneous group in the $GF[p^n]$ defined by a quadratic invariant is investigated. The case when $p > 2$ is considered in the former and that when $p = 2$ in the latter of these chapters. By employing the theory of compounds, developed in the preceding chapter, the investigations, especially when $p > 2$, have been presented in a much simpler manner than in the original papers. Chapter VIII is followed by twelve exercises with a number of suggestions, covering the text of the first eight chapters of this part.

The next three Chapters (IX–XI) are devoted respectively to linear groups with certain invariants whose degree exceeds 2, canonical form and classification of linear substitutions, and operators and cyclic subgroups of the simple linear fractional group $LF(3, p^n)$. In Chapter XII the subgroups of the linear fractional group $LF(2, p^n)$ are studied. The first statement of the last paragraph of this chapter is evidently incorrect. For instance, the icosahedral group may be represented as a transitive substitution group of degree 12, but it does not contain a complete system of 12 conjugate subgroups. If a simple group is represented on the smallest possible number of letters it must be primitive, and a primitive group of degree N must always contain a complete system of N conjugate subgroups; but this is not necessarily true of imprimitive groups of degree N even if they are simple.

The developments of Chapters XIII and XIV are quite different from those which precede. In the former a number

* Dickson, *Math. Annalen*, vol. 52 (1899), p. 561.

† Dickson, *BULLETIN*, vol. 5 (1898), p. 120; *Proc. Lond. Math. Soc.*, vol. 30 (1898), p. 70.

of theorems in regard to abstract groups are developed. Beginning with the well known theorems of Moore with respect to the abstract definitions of the alternating and the symmetric groups and the classic theorem due to Hamilton and Dyck in regard to the abstract definition of the icosahedral group, the author develops a number of other theorems with respect to the generational relations of known groups. Some of these are employed in the study of the group of the equation for the 27 straight lines on a general surface of the third order, which is the subject of Chapter XIV. The treatment of this subject seems to be especially commendable.

The study of linear groups has thus far been the most fruitful in increasing our knowledge of simple groups. As in Jordan's *Traité des Substitutions*, so in the present work a great deal of attention is given to the factors of composition of the groups under consideration. In this way a number of systems of simple groups are established and the closing chapter is fittingly devoted to a summary of the known simple groups. With a few exceptions all of these groups are included in one or more known infinite systems. It may be observed that the simple group of order 7920 cannot be represented on less than 11 letters so that 9 should be replaced by 11 in the list on page 309.

The present work seems to be the first separate volume on the subject to which it is devoted. The treatment of the first part has much in common with Chapters III and IV of Serret's *Algèbre supérieure*, volume 2 (1885), while the second part seems to have been mostly influenced by Jordan's *Traité des Substitutions* (1870). The author's thorough acquaintance with his subject, and his experience as a writer, have enabled him to present a somewhat abstract subject in an unusually attractive manner. It may be hoped that the book will do much to create a wider interest in these fields of higher algebra. While the generality of treatment calls for considerable maturity on the part of the reader, yet not much special knowledge is presupposed.

G. A. MILLER.

STANFORD UNIVERSITY.

BUCKINGHAM'S THERMODYNAMICS.

Theory of Thermodynamics. By EDGAR BUCKINGHAM, Ph.D.
New York, The Macmillan Company, 1900. xi + 205 pp.
Price, \$1.90.

OLDFASHIONED, or, rather, elementary, thermodynamics deals mainly with the relations of heat and mechanical energy. It considers purely physical changes, including changes from any one to any other of the three states, solid, liquid and gaseous, but does not undertake to discuss chemical changes or such chemico-physical changes as occur in solution. Accordingly, from the purely mathematical point of view, elementary thermodynamics is shamefully simple. What possibilities of interest for the mathematician can be found in a beggarly array of only five variables, p , v , T , ϵ and η , subject to one characteristic equation and two general laws, so that only two of them at a time are capable of arbitrary variation? But from the standpoint of physical interpretation and application the study of thermodynamics, even in its elements, presents respectable difficulties, and is likely to be regarded by the beginner, however mathematical he may be, as something of a mystery.

On the other hand, when we undertake to deal with chemical changes, and find ourselves confronted by " i phases" of matter, each phase containing "a mixture of K substances," the variables, "normal" and "inverse," "internal" and "external," independent and dependent, begin to manœuvre in regiments at the will of the commander, and the born mathematician finds himself in his proper element.

The book before us is intended for the help of those not very rare individuals who have some difficulty with both the physics of elementary thermodynamics and the mathematics of its broader generalizations. It is not for the beginner, although it deals with the very elements and discusses the two general laws at length. It is too circumspect, too precise, too thorough-going, in its examination for the student who is taking his first look at the field with which it has to do. It is rather for him who, after some months of growing acquaintance with the elementary facts and laws of thermodynamics, finds himself in doubt as to whether he fully understands, for example, the

nature of a reversible cycle or the theory of the "porous plug" experiments by means of which the "absolute thermodynamic scale of temperature" was brought to concrete existence. The very great care which the author uses in examining the premises of his science is shown by the following quotations. "It is usual to speak of this temperature [the temperature indicated by a thermometer in a mixture of liquids or gases] of the mixture as being also the temperature of the separate components, and this mode of expression does not lead us into any practical difficulties, although it has evidently no justification *a priori*" (page 2); "Our first idea of heat is that of something which increases the temperature of a body when added to it and decreases the temperature when taken away [a footnote here excludes melting, evaporation, etc., from immediate consideration]. The first addition which we make to this conception is the *assumption*, that *when a given body cools through a definite interval, the quantity of heat it gives out is always the same*, regardless of what becomes of the heat after it leaves the body in question" (page 9). Such care is admirable and in the main reassuring, even when it results in stamping as an assumption what most readers would be likely to take as an axiom. In at least one case, however, the author's caution seems to the reviewer too great; namely, on pages 9 and 10, where he raises the question, "Are two quantities of heat, which are equal to a third, equal to each other?" and disposes of it by the "definition," "*that quantities of heat which are equal to the same quantity are equal to one another.*" It is true that the author has been feeling his way to the conception that heat is "a quantity in the ordinary sense," but after he has got so far as to say that two quantities of heat are each equal to a third, it is too late for the need of any argument or definition to the effect that these two quantities are equal to each other.

The following definition, "Any process, of which the direction may be reversed by infinitely small modifications of the outside actions, is called a REVERSIBLE PROCESS," and, in general, everything that the author has to say about reversible or non-reversible processes is luminous, precise, and much to be commended.

In a word, even the reader who does not care for the chemical applications of thermodynamics, and who has no stomach for generalized coördinates, will find much in this book that he

can profit by, though it is not written for him alone or for him primarily.

But what of him whose ambition is to go further? For him there is an excellent brief chapter on The Conditions of Thermodynamic Equilibrium, and another on Thermodynamic Potentials and Free Energy. These are preceded by the necessary disquisition on the relations and functions of the indefinitely large number of variables which may be involved, and they are followed by a chapter of applications in which are discussed the electromotive force of a reversible galvanic cell, the equilibrium of phases of a single substance (triple point, etc.), and the phase rule of Gibbs. The discussion of the last subject disclaims the intention of being "either complete or rigorous"; but as an introduction "'tis enough, 'twill serve." There is, probably, no other book so well suited as this to the needs of him who is making preparation for an attempt to explore those tremendous abysses of thought, where reigns that condition of supernal calm known as the Equilibrium of Heterogeneous Substances.

But, alas! like the fellow who had never learned to read, and who found that no spectacles, however well contrived, would enable him to read, the reviewer gazes in vain through the medium of Dr. Buckingham's book at many a passage like the following in the famous third volume of the *Transactions of the Connecticut Academy*, "the stability of any phase in regard to continuous changes depends upon the same conditions in regard to the second and higher differential coefficients of the density of energy regarded as a function of the density of entropy and the densities of the several components, which would make the density of energy a minimum, if the necessary conditions in regard to the first differential coefficients were fulfilled."

Through *Nature to Gibbs* should be the watchword of every devout thermodynamician, and if Dr. Buckingham has not succeeded in making the way easy, the failure is not his fault.

EDWIN H. HALL.

CAMBRIDGE, MASS.,
July 23, 1902.

NOTES.

CERTAIN numbers of the *Bulletin of the New York Mathematical Society* are now nearly out of print. The committee of publication is especially desirous of procuring copies of volume 2, nos. 6 and 7, and volume 3, nos. 3 and 9, and is willing to pay one dollar per copy for these numbers. The committee respectfully requests the members of the Society to look over their files and to notify the committee of any duplicates of the numbers mentioned with which they are willing to part. The Society receives a considerable revenue from the sales of sets of the BULLETIN and is often able to secure desirable exchanges of back volumes of other mathematical journals in return for its own. These two interests will be seriously affected as soon as the present small supply of the numbers in question is exhausted, unless the committee of publication is able to replace them.

THE twelfth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held in the Ryerson Physical Laboratory of the University of Chicago, on Friday and Saturday, January 2 and 3, 1903. Titles and abstracts of papers should be in the hands of the secretary of the section, Professor T. F. Holgate, 617 Hamline St., Evanston, Ill., at least two weeks before the meeting.

THE second regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY will be held in room 21, North Hall, of the University of California, on Saturday, December 20, 1902. Titles and abstracts of papers should be in the hands of the secretary of the section, Professor G. A. Miller, Stanford University, at least two weeks before the meeting.

A NEW edition of the Annual Register of the AMERICAN MATHEMATICAL SOCIETY is now in preparation and will be issued in January next. Forms for furnishing necessary information have been sent to each member. Those who have not already responded to the circular are requested to do so at once.

THE October number (volume 4, number 1, second series) of the *Annals of Mathematics* contains: "The geodesic lines

on the anchor ring," by G. A. BLISS; "Proof of a theorem concerning isosceles triangles," by H. F. BLICHFELDT; "An elementary exposition of Frobenius's theory of group characters and group determinants," by L. E. DICKSON; "Communication concerning Mr. Ransom's mechanical construction of conics," by E. V. HUNTINGTON.

CAMBRIDGE UNIVERSITY.—Advanced mathematical courses for the current academic year are announced as follows:—

Michaelmas term, 1902.—By Professor Sir G. G. STOKES: Hydrodynamics, including viscosity, three hours.—By Professor A. R. FORSYTH: Differential geometry; general theory of curves and surfaces, three hours; Fourier's and other expansion theorems, two hours.—By Professor G. H. DARWIN: Orbits of planets; Figures of equilibrium of rotating fluid, three hours.—By Professor Sir R. S. BALL: Planetary theory, three hours.—By Dr. E. W. HOBSON: Spherical and cylindrical harmonics, three hours.—By Mr. J. LARMOR: Electrodynamics, three hours.—By Mr. H. F. BAKER: Elementary theory of functions, three hours.—By Mr. H. M. MACDONALD: Waves (especially waves of light), three hours.—By Mr. H. W. RICHMOND: Analytic geometry of curves, three hours.—By Mr. G. T. WALKER: The electromagnetic field, three hours.—By Mr. G. B. MATHEWS: Algebraic functions (elementary).—By Mr. A. N. WHITEHEAD: Application of symbolic logic to Cantor's theory of aggregates, three hours.—By Mr. J. H. GRACE: Invariants and geometric applications, three hours.

Lent term, 1903.—By Professor Sir G. G. STOKES: Physical optics, three hours.—By Professor A. R. FORSYTH: Differential geometry; general theory of curves and surfaces (continued), three hours.—By Professor G. H. DARWIN: Outlines of dynamical astronomy, three hours.—By Dr. E. W. HOBSON: Sound and vibrations, three hours.—By Mr. J. LARMOR: Electrodynamics, with optical and thermodynamic applications, three hours.—By Mr. H. W. RICHMOND: Analytic geometry of three dimensions, projective properties, three hours.—By Mr. H. F. BAKER: Theory of functions.—By Mr. H. M. MACDONALD: Hydrodynamics, three hours.—By Mr. G. B. MATHEWS: Theory of algebraic numbers, three hours.—By Mr. R. A. HERMAN: Hydrodynamics.—By Mr. A. BERRY: Elliptic functions.—By Mr. G. T. WALKER: The electromagnetic theory of light, three hours.—By Mr. G. T. BENNETT: Linear and quadratic complexes, three hours.—By Mr. E. T. WHIT-

TAKER: The theory of definite integrals, two hours. By **Mr. J. H. GRACE**: Invariants and geometric applications (continued), three hours.—By **Mr. HUDSON**: Geometric theory of ordinary differential equations.

Easter term, 1903.—By **Professor Sir R. S. BALL**: Perturbation of cometary orbits, three hours.—By **Mr. W. L. MOLLISSON**: Theory of potential and electrostatics, three hours.—By **Mr. A. N. WHITEHEAD**: Non-euclidean geometry.

Long vacation, 1903.—By **Professor Sir R. S. BALL**: Applications of modern geometry to dynamics, three hours (short course).—By **Mr. E. T. WHITTAKER**: General dynamics, with applications to astronomy and thermodynamics, three hours.

THE Belgian Academy at Brussels, announces as its prize subject for 1903, the following:

“An important contribution to the study of mixed forms in any number of variables, with the application of the results to the geometry of any spaces.”

THE London Mathematical Society has awarded its De Morgan Medal for 1902 to **Professor A. G. GREENHILL**, of the Ordnance College, Woolwich.

ON December 15, the University of Klausenberg will celebrate the hundredth anniversary of the birth of **JOHANN BOLYAI**.

PROFESSOR K. HENSEL, of the University of Berlin, has been called to a professorship of mathematics at the University of Marburg.

MR. ALFRED T. DE LURY, M.A., formerly lecturer in mathematics in the University of Toronto, has been appointed associate professor in the same department. **DR. J. C. FIELDS** has been appointed special lecturer in mathematics in the University of Toronto.

PROFESSOR L. E. DICKSON, of the University of Chicago, is now associated with **Professor B. F. FINKEL**, of Drury College, as editor of the *American Mathematical Monthly*. **Mr. J. M. COLAW**, formerly associate editor of the *Monthly*, has retired from the editorial staff.

DR. J. W. MILLER has been appointed instructor in mathematics and astronomy at Lehigh University.

DR. C. N. HASKINS has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

PROFESSOR XAVIER ANATOMARI, editor of the *Nouvelles annales de mathématiques*, died at Paris, on June 9, 1902.

PROFESSOR ERNST SCHRÖDER, of the department of mathematics in the Polytechnic school in Carlsruhe, died on June 17, 1902.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BETTI (E.). Opere matematiche, pubblicate per cura della R. Accademia dei Lincei. (In 2 volumes.) Vol. I. Milano, Hoepli, 1902. 4to. Fr. 25.00

COMPTE RENDU du deuxième congrès international des mathématiciens, tenu à Paris, du 6 au 12 août 1900. Procès-verbaux et communications, publiés par E. Duporcq, secrétaire général du congrès. Paris, Gauthier-Villars, 1902. 8vo. 456 pp. Fr. 16.00

DUPORCQ (E.). See COMPTE RENDU.

DZIWIŃSKI (P.). Wykłady matematyki, Kurs I. Zasady geometrii analitycznej i analizy wyższej, tom I. Wstęp do geometrii i teoria wyznaczników. Początki analizy wyższej i zasady rachunku różniczkowego. Lwów, 1902. 4°. 19 + 928 pp. M. 30.00

GODEFROY (M.). Théorie élémentaire des séries (limites; séries à termes constants; séries à termes variables; fonction exponentielle; fonctions circulaires; fonction gamma). Avec une préface de L. Sauvage. Paris, Gauthier-Villars, 1903. 8vo. 8 + 266 pp. Fr. 8.00

HEDRICK (E. R.). On the sufficient conditions in the calculus of variations. 8vo. (*Bulletin of the American Mathematical Society* (2) 9, pp. 11-24.)

IGUREIDE (J. F.). Nature harmonique de l'espace; traduit de l'espagnol. Barcelone, Giro, 1902. 8vo. 245 pp.

MONTCEUIL (M. DE). Sur une classe de surfaces. (Thèse.) Paris, Gauthier-Villars, 1902. 4to. 83 pp.

RÉPERTOIRE bibliographique des sciences mathématiques. 2e série. Fiches 1001 à 1100. Paris, Gauthier-Villars, 1901. 18mo. Fr. 2.00

SAPOLSKY (L.). Ueber die Theorie der relativ-Abelschen cubischen Zahlkörper. (Diss.) Leipzig, Teubner, 1902. 8vo. 7 + 481 + 6 pp., 35 plates. M. 6.00

SAUVAGE (L.). See GODEFROY (M.).

SIMON Y MAYORGA (J.). Caracteres de irracionalidad de los números enteros. Trabajo de investigación. Salamanca, Calón, 1902. 8vo. 20 pp. Fr. 2.00

- STUYVAERT. Etude de quelques surfaces algébriques engendrées par des courbes du second et du troisième ordre. (Thèse.) Gand, Hoste, 1902. 8vo. 74 pp.
- WEINNOLDT (E.). Leitfaden der analytischen Geometrie. Leipzig, Teubner, 1902. 8vo. 6 + 80 pp. Cloth. M. 1.60
- WHITTAKER (E. T.). A course of modern analysis. An introduction to the general theory of infinite series and of analytic functions, with an account of the transcendental functions. Cambridge, Macmillan & Bowes, 1902. 8vo. Cloth. 12s. 6d.

II. ELEMENTARY MATHEMATICS.

- ARRIAGA (M.). Geometría elemental y superior, con ejercicios de cálculo y de dibujo lineal. Barcelona, Araluce [1902]. 8vo. 241 pp.
- BERGMEISTER (H.). Geometrische Formenlehre für Mädchen-Lyzeen. Teil 1: für die I. und II. Klasse. Wien, Deuticke, 1902. 8vo. 4 + 88 pp. Cloth. M. 1.50
- BETTAZZI (R.). Aritmetica razionale ad uso dei ginnasi. Torino, Tipografia Salesiana, 1902. 8vo. 180 pp. Fr. 2.00
- BRIOT (C.). Tratado de algebra elemental y superior. Traducción de C. Sebastián y B. Portuondo, corregida, revisada y ampliada por B. Portuondo. Madrid, Hernando, 1901. 8vo. 732 pp. Fr. 14.00
- BRUNN (J.). Vierstellige Logarithmen, für den Schulgebrauch zusammengestellt. Münster, Aschendorff, 1902. 8vo. 18 pp. M. 0.25
- CERCIGNANI (E.). Riassunto d'algebra; appunti ad uso degli alunni della scuola professionale Leonardo da Vinci, anno scolastico 1901-02. Firenze, Ramella, 1902. 8vo. 54 pp.
- DEAKIN (R.). Matriculation algebra; being the tutorial algebra, elementary course. 2d edition. London, Clive, 1902. 12mo. 464 pp. Cloth. (University tutorial series.) 3s. 6d.
- DUPORT (J. B.). Lehrbuch der Arithmetik für die erste Klasse der Mädchen-Lyzeen. Wien, Deuticke, 1902. 8vo. 3 + 82 pp. Cloth. M. 1.20
- ELLIOTT (J.). Elementary geometry. London, Sonnenschein, 1902. 12mo. 280 pp. Cloth. 4s.
- FRANKLAND (W. B.). The story of Euclid. New York, Wessels, 1902. [Imported.] 16mo. 176 pp. Portrait. Cloth. \$0.75
- GAMBIOLI (D.). Breve sommario della storia delle matematiche, colle due appendici sui matematici italiani e sui tre celebri problemi geometrici dell'antichità, ad uso delle scuole secondarie. Bologna, Zanichelli, 1902. 16mo. 241 pp. Fr. 3.00
- GERRISH (C.) and WELLS (W.). The beginner's algebra. Boston, Heath, 1902. 12mo. 10 + 151 pp. Cloth. \$0.50
- HOÛEL (J.). Recueil de formules et de tables numériques. 3e édition. Paris, Gauthier-Villars, 1901. 8vo. 71 + 64 pp. Fr. 4.50
- HUPE (A.). See MÜLLER (H.).
- J. (F.). Éléments de géométrie, comprenant des notions sur les courbes usuelles, un complément sur le déplacement des figures et de nombreux exercices. Paris, Poussielgue [1902]. 16mo. 12 + 525 pp. (Cours de mathématiques élémentaires.)

- JONES (A. C.). *Beginnings of trigonometry.* London, Longmans, 1902. 12mo. 152 pp. Cloth. 2s.
- MÜLLER (H.). *Die Mathematik auf den Gymnasien und Realschulen, für den Unterricht dargestellt.* Ausgabe B: Für reale Anstalten und Reformschulen, unter Mitwirkung von A. Hupe. Teil 2: Die Oberstufe; Lehraufgabe der Klassen Ober-Sekunda und Prima. Abteilung II: Synthetische und analytische Geometrie der Kegelschnitte; darstellende Geometrie. 2te Auflage, Leipzig, Teubner, 1902. 8vo. 8 + 178 pp., 2 plates. Cloth. M. 2.40
- NITSCH (J.). *Lehr- und Übungsbuch der Arithmetik für die I. und II. Gymnasialklasse.* Wien, Deuticke, 1902. 8vo. 3 + 131 pp. Cloth. M. 1.80
- PORTUONDO (B.). See BRIOT (C.).
- SÁNCHEZ RAMOS (E.) y SABRAS Y CAUSAPE (T.). *Elementos de trigonometría rectilínea.* 2a edición. Salamanca, Núñez [1902]. 8vo. 56 pp.
- SCHULTZE (A.) and SEVENOAK (F. L.). *Plane geometry.* New York, Macmillan, 1902. 12mo. 11 + 233 pp. Cloth. \$0.80
- SEBASTIAN (C.). See BRIOT (C.).
- SEGALL (S.). *Praktische Anleitung zur Erlernung des Ansatzes von Gleichungen.* Leipzig, Paul, 1902. 32mo. 52 pp. M. 0.10
- SEVENOAK (F. L.). See SCHULTZE (A.).
- SMITH (H. W. C.). *First principles of ratio and proportion, and their application to geometry.* London, Macmillan, 1902. 12mo. 1s.
- VALLERÉY (J.). *Arithmétique; algèbre; trigonométrie.* Ouvrage rédigé conformément aux programmes des examens. 2e édition, entièrement revue et augmentée. Paris, Challamel, 1902. 8vo. 8 + 484 pp. (Bibliothèque des capitaines de commerce et des candidats aux examens de la marine marchande.)
- VISALLI (P.). *Algebra.* Livorno, Giusti, 1902. 16mo. 4 + 160 pp. (Biblioteca degli studenti: riassunti per tutte le materie d'esame nei licei, ginnasi, istituti tecnici, ecc., vol. 66-67.) Fr. 1.00
- WELLS (W.). See GERRISH (C.).
- WIENECKE (E.). *Geometrie für Volksschulen mit circa 400 Übungsaufgaben.* 2te Auflage. Berlin, Oehmigke, 1902. 8vo. 2 + 77 pp. M. 0.45

III. APPLIED MATHEMATICS.

- BOLTZMANN (L.). *Leçons sur la théorie des gaz; traduites par A. Gallotti. Avec une introduction et des notes de M. Brillouin.* Première partie. Paris, Gauthier-Villars, 1902. 8vo. 19 + 204 pp. Fr. 8.00
- BOX (T.). *Practical treatise on strength of materials, including their elasticity and resistance to impact.* 4th edition. London, Spon, 1902. 8vo. 538 pp. 12s. 6d.
- BRAGSTAD (O. S.). *Theorie des rotierenden Feldes mit Anwendung auf die Bestimmung des Stromdiagrammes der asynchronen Maschinen.* (Habilitationsschrift, Karlsruhe, 15. Juni 1901.) Stuttgart, Deutsche Verlagsgesellschaft, 1902. 8vo. 4 + 71 pp.

- BRILLOUIN. See BOLTZMANN (L.).
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The following dates have been fixed for the meetings of the Society :

	A. M.		A. M.
Sat., Oct. 25, 1902, 11:00		Sat., Feb. 28, 1903, 11:00	
M.-Tu., Dec. 29-30, " "		April 25, " "	

The Meeting of December 29-30 will be the Annual Meeting of the Society for the Election of Officers and Delivery of the Presidential Address.

The Chicago Section will meet at the University of Chicago, January 2-5, 1903.

The San Francisco Section will meet at the University of California, December 20, 1902.

BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

CONTINUATION OF THE BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

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F. N. COLE A. ZIWET D. E. SMITH F. MORLEY

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THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 25, 1902. About forty-five persons attended the morning and afternoon sessions, including the following thirty-five members of the Society :

Professor Maxime Bôcher, Professor Joseph Bowden, Dr. J. E. Clarke, Professor F. N. Cole, Dr. W. S. Dennett, Professor A. M. Ely, Dr. William Findlay, Professor T. S. Fiske, Miss Ida Griffiths, Dr. E. R. Hedrick, Dr. G. W. Hill, Dr. A. A. Himowich, Dr. E. V. Huntington, Dr. S. A. Joffe, Dr. Edward Kasner, Dr. G. H. Ling, Mr. H. B. Mitchell, Dr. I. E. Rabinovitch, Professor J. K. Rees, Mr. C. H. Rockwell, Dr. P. L. Saurel, Miss I. M. Schottenfels, Professor C. A. Scott, Professor D. E. Smith, Dr. Virgil Snyder, Dr. H. F. Stecker, Miss Mary Underhill, Professor E. B. Van Vleck, Professor L. A. Wait, Mr. H. E. Webb, Professor J. B. Webb, Professor A. G. Webster, Miss E. C. Williams, Dr. R. G. Wood, Professor R. S. Woodward.

Vice-President Professor Maxime Bôcher presided at the morning session, Ex-President Professor R. S. Woodward at the afternoon session. The Council announced the election of the following persons to membership in the Society : Professor Sir R. S. Ball, Cambridge University, England ; Dr. Otto Dunkel, Wesleyan University, Middletown, Conn. ; Mr. W. H. Osborne, Purdue University, Lafayette, Ind. ; Professor H. L. Rietz, Butler College, Indianapolis, Ind. ; Professor G. H. Scott, Yankton College, Yankton, So. Dak. ; Professor J. N. Van der Vries, University of Kansas, Lawrence, Kan. ; Professor B. F. Yanney, Mount Union College, Alliance, Ohio ; Mr. W. H. Young, Cambridge University, England. Seven applications for membership in the Society were received.

A list of nominations to be placed on the official ballot for the annual election of officers and other members of the Council was reported. A committee consisting of the Secretary and Professors W. F. Osgood, Oskar Bolza, James Pierpont, and

G. A. Miller was appointed to make arrangements for the next summer meeting and colloquium of the Society.

The following papers were presented at this meeting :

(1) DR. E. R. HEDRICK : "On the foundations of mechanics" (preliminary communication).

(2) DR. E. V. HUNTINGTON : "Definitions of a commutative group by independent postulates."

(3) PROFESSOR PETER FIELD : "On the infinite branches of plane curves which have no point singularities."

(4) DR. EDWARD KASNER : "The apolarity of double binary forms."

(5) PROFESSOR MAXIME BÔCHER : "An application of the Riemann-Darboux generalization of Green's theorem."

(6) PROFESSOR MAXIME BÔCHER : "Note on Laplace's equation."

(7) DR. VIRGIL SNYDER : "On the quintic scrolls having three double conics."

(8) MISS I. M. SCHOTTENFELS : "Note on the types of groups of order p^n every element of which, except identity, is of order p " (preliminary communication).

(9) DR. L. P. EISENHART : "Surfaces referred to their lines of length zero."

(10) PROFESSOR L. E. DICKSON : "Three sets of generational relations defining the abstract simple group of order 504."

(11) PROFESSOR L. E. DICKSON : "Generational relations defining the abstract simple group of order 660."

(12) DR. G. H. LING : "The approximate representation of functions by means of functions defined by quadratic equations."

(13) DR. C. N. HASKINS : "On the invariants of differential forms of degree higher than two."

Dr. Haskins's paper was presented to the Society through Professor Bôcher. In the absence of the authors, Professor Field's paper was read by Dr. Snyder, that of Dr. Haskins by Professor Bôcher, and the papers of Dr. Eisenhart and Professor Dickson were read by title.

The papers of Dr. Snyder, Dr. Eisenhart and Professor Dickson will appear in the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In Dr. Hedrick's preliminary report, an attempt is made to establish a logical system of axioms, sufficient for the deriva-

tion of the elementary theorems in the mechanics of a system of particles. The ideas of space and time are made fundamental, space being taken as that configuration of points, lines and planes defined by Hilbert's set of axioms in his *Grundlagen der Geometrie*, while time is defined as a parameter, in terms of which the equations of any curve in space may so be expressed that these equations represent the motion of the particle in the ordinary sense. Axioms are then laid down which fix (or define) the ideas "before," "after," etc.

Mass is now defined in terms of time and space, on the basis of Newton's law of universal gravitation, previous axioms having required a certain form of motion in the case when two particles are isolated in space. Force is then defined as usual, and the remaining axioms necessary to the development of the elementary theorems are laid down. This preliminary report, is intended to be a forerunner of a more complete discussion, and the purpose of presenting it at this time is to invite suggestion and criticism.

2. The definitions of an abelian (commutative) group given in Dr. Huntington's paper are simple modifications of the author's definitions of a general group published in the last volume of the *BULLETIN*.^{*} According to the first definition, a set of elements, with a rule of combination $\circ$, will be an abelian group when the following three postulates are satisfied: 1° $a \circ b = b \circ a$, whenever a, b and $b \circ a$ belong to the set. 2° $(a \circ b) \circ c = a \circ (b \circ c)$, whenever $a, b, a \circ b, b \circ c$ and $a \circ (b \circ c)$ belong to the set. 3° For every two elements a and b there is an element x in the set, such that $a \circ x = b$. The second definition involves four postulates. The independence of the postulates of each set is established both for finite and for infinite groups. The paper will appear in the *Transactions*.

3. Professor Field's paper considers the form of sextic curves which have no point singularities and which have two essential infinite branches. It is shown that such curves may be unipartite or multipartite. The form and equation of a multipartite curve of degree $3n$ with no point singularities and with n infinite branches is thus obtained, and also the form and equation of a non-singular unipartite curve of degree $12n$ with

^{*} Cf., on this subject an article by Professor E. H. Moore in the *Transactions* for October, 1902.

4n essential infinite branches. The question as to the maximum number of essential infinite branches which a curve without point singularities may possess is then considered, it being shown that such a curve may have $n - 4$ essential infinite branches and that this is the maximum number.

4. In a paper published in the first volume of the *Transactions* Dr. Kasner established a connection between the theories of double binary and quaternary forms. In the present paper the author applies this connection to certain questions as to apolarity. A double binary form $f_{nm} = f(x_1, x_2; y_1, y_2)$ of the n th degree in x_1, x_2 and the m th degree in y_1, y_2 , may be interpreted as the general algebraic curve of species (n, m) upon a quadric surface Q . When the two degrees are equal the curve is a complete intersection curve, and there corresponds to the form f_{nn} a unique surface of n th order F_n and also a unique surface of n th class ϕ_n . If two forms f_{nn} and $f'_{n'n'}$, where $n \equiv n'$, are apolar, then every surface of the n' th order through the curve $f'_{n'n'} = 0$ is apolar to the surface ϕ_n which corresponds to f_{nn} ; conversely, if ϕ_n is apolar to any surface of n' th order through $f'_{n'n'} = 0$, then the double binary forms are apolar. This theorem, applied to the case where one of the forms is bilinear, reduces the problem of constructing all the curves on a quadric surface which are apolar to a conic to the corresponding problem of plane geometry. The discussion of apolar relations with respect to forms f_{nm} where $m \neq n$ occupies the final part of the paper. The method is based on the introduction of certain systems of polar forms and the related systems of surfaces.

5. The form of Green's theorem here used will be found in the *Encyclopädie*, II, A7c, pages 513, 514. Using the notation and terminology there explained, it is here proved that if throughout a certain region S of the (x, y) -plane the partial differential equation $L(u) = 0$ belongs to the elliptic type, and if u is a solution of this equation analytic throughout S , and if a function v exists continuous and having continuous first and second partial derivatives throughout S and satisfying the inequality $AM(v) \leq 0$, then u cannot vanish at all points of a closed curve in S , unless it vanishes identically.

6. It is well known that if $f_0(y)$ and $f_1(y)$ are two functions which can be developed in power series about the point $y = 0$,

there exists one and only one solution $u(x, y)$ of Laplace's equation in two dimensions which, when $x = 0$, reduces to $f_0(y)$ while its partial derivative with regard to x reduces to $f_1(y)$, and that this function $u(x, y)$ can itself be developed in a power series about the point $x = 0, y = 0$. The chief theorem proved in the present paper is that if the series f_0 and f_1 both converge when $|y| < r$, the double power series for $u(x, y)$ converges within the square whose vertices are the points $(\pm r, 0), (0, \pm r)$.

8. Miss Schottenfels's note on types of groups of order p^n , all of whose elements (I excluded) are of order p , contains a proof of the existence of a group of indices (modulo p) where p is a prime, and the generational definition of the abelian group G_{p^n} every element of which is of prime order p .

12. Dr. Ling's paper contains a treatment of a problem of a type suggested by Hermite, viz., that of the approximate representation of a function by means of algebraic functions. Padé, after showing the relation of the representation of functions by means of power series to the general problem, treated the case of functions defined by linear equations. In the present paper is considered the case of functions defined by quadratic equations of the type $y^2 + Qy + P = 0$, where Q and P are polynomials. After a definition of *reduced function* has been given, the paper treats of the determination of such functions in the general case and of the order of approximation to a specified function which is obtained in each case. The results obtained are generalizations of the two theorems proved by Padé.

13. Dr. Haskins's paper is an extension of the work in the author's paper on the invariants of quadratic differential forms to forms of higher degree. The methods used are the same, but the difficulties of the problem are much less. The result is as follows: The number of invariants of order μ for the general homogeneous differential form of degree m in n variables is

$$(n, \mu) \left\{ (n, m) - \frac{n(n + \mu)}{\mu + 1} \right\}, \quad (n, m) = \frac{(n + m - 1)!}{(n - 1)! m!}$$

The case of simultaneous invariants of several forms is also considered.

F. N. COLE.

COLUMBIA UNIVERSITY.

SERIES WHOSE PRODUCT IS ABSOLUTELY CONVERGENT.

BY PROFESSOR FLORIAN CAJORI.

(Read before Section A of the American Association for the Advancement of Science, Pittsburg, July 1, 1902.)

§ 1. That two absolutely convergent series yield an absolutely convergent product was first shown by Cauchy.* About three quarters of a century later Alfred Pringsheim pointed out that an absolutely convergent product may result also from the multiplication of a conditionally convergent or even a divergent series by an absolutely convergent series.† That the product of two conditionally convergent series, or of a conditionally convergent series and a divergent series, or of two divergent series, may be absolutely convergent was first made public by the present writer.‡ Thereupon Alfred Pringsheim treated the subject from a more general point of view and, by very simple methods, showed that the property in question is typical of certain classes of series.§ The present writer developed a new class of series possessing this property, demonstrated the validity of the fundamental laws of algebra in the multiplication of infinite series, and generalized a theorem of Abel on the multiplication of series.|| In the present article we aim to generalize some of the results previously obtained relating to absolutely convergent products of two or more series.

§ 2. In this investigation we shall start with an absolutely convergent series and determine pairs of series which are factors of the assumed series. Given the absolutely convergent series

$$U = \sum_{n=0}^{\infty} u_n, \quad (1)$$

in which

$$u_n = \sum_{r=0}^r c_r e_r c'_{n-r} e_{n-r},$$

* *Analyse Algébrique*, 1821, page 147.

† *Math. Annalen*, vol. 21 (1883), pp 357-359.

‡ *Transactions of the American Mathematical Society*, vol. 2, pp 25-36, January 1901; *Science*, new series, vol. 14, p. 395, September 13, 1901.

§ *Transactions of the American Mathematical Society*, vol. 2, pp. 404-412, October, 1901.

|| *BULLETIN*, 2d series, vol. 8, pages 231-236, March, 1902.

e_r is the general term of any absolutely convergent series, say

$$e_r = \frac{1}{(z+r)[\log(z+r)]^\lambda},$$

$\lambda > 1$, and c_r, c'_r are constants either real or complex. That series (1) is absolutely convergent becomes evident, if we observe that it is the product formed according to Cauchy's multiplication rule, of the two absolutely convergent series

$$\sum_{r=0}^{\infty} c_r e_r \quad \text{and} \quad \sum_{r=0}^{\infty} c'_r e_r.$$

Let us assume

$$a_{r-s} - a_r = c_r e_r, \quad (2)$$

$$h_r (b_{r-t} - b_r) = c'_r e_r,$$

where s and t are positive integers, a and b are real or complex numbers and h_r is an odd or an even power of -1 . Let it be agreed that a and b cannot have negative subscripts; in other words, that $a_{-x} = b_{-x} = 0$. We have

$$u_n = \sum_{r=0}^{r=n} c_r e_r \cdot c'_{n-r} e_{n-r} = \sum_{r=0}^{r=n} h_{n-r} (a_{r-s} - a_r) (b_{n-r-t} - b_{n-r}).$$

If we perform the indicated multiplications and collect the coefficients of a_r , we obtain

$$u_n = \sum_{r=0}^{r=n} a_r (-h_{n-r} b_{n-r-t} + h_{n-r} b_{n-r} + h_{n-r-s} b_{n-r-s-t} - h_{n-r-s} b_{n-r-s}). \quad (3)$$

If we assume $h_s = -h_{s+t}$, we have

$$u_n = \sum_{r=0}^{r=n} (a_{r-t} + a_r) (h_{n-r} b_{n-r} - h_{n-r-s} b_{n-r-s}). \quad (4)$$

It will be noticed that if, in the two terms in (3) which involve the factors a_r and a_{r-t} , respectively, we remove the parentheses, we obtain eight terms which are distributed among three terms

of the series in (4), namely the three terms which involve, respectively, the parentheses

$$(a_{r-2t} + a_{r-t}), \quad (a_{r-t} + a_r), \quad (a_r + a_{r+t}).$$

From the inspection of series (4) we readily see that the series (1) may be considered to be the product of the following two series :

$$\sum_{r=0}^{\infty} (a_{r-t} + a_r) \equiv a_0 + a_1 + \dots + a_{t-1} + (a_0 + a_t) \\ + (a_1 + a_{t+1}) + \dots \quad (5)$$

and

$$\sum_{r=0}^{\infty} (h_r b_r - h_{r-t} b_{r-t}) \equiv h_0 b_0 + h_1 b_1 + \dots + h_{t-1} b_{t-1} \\ + (h_t b_t - h_0 b_0) + \dots \quad (6)$$

According to the condition $h_r = -h_{r+t}$, we are permitted to choose any sign we please for any t consecutive factors h_r . After such a choice has been made, the signs represented by any of the other factors h_r are determined.

Since relations (2) are the only conditions imposed upon the values of a and b , it is possible to choose these values so that each of the series (5) and (6) is absolutely convergent, conditionally convergent, or divergent. Thus, if $|a_r|$ is of the order of magnitude e_r , series (5) is absolutely convergent; if

$$|a_{r-t} + a_r| \cong \frac{1}{(r+2) \log(r+2)},$$

but approaches the limit zero as r increases indefinitely and if $(a_{r-t} + a_r)$ is opposite in sign to and has greater numerical value than $(a_{r-t+1} + a_{r+1})$, then (5) is conditionally convergent; if $|a_{r-t} + a_r|$ does not approach the limit zero for all values of r , as r increases indefinitely, the series (5) is divergent. Similarly for (6). Yet in every case, the product of (5) and (6) is absolutely convergent.

Since, so far as we know, no two divergent series with *complex* terms have before been given, whose product is absolutely convergent, it may be well to construct a special example. Let

$a_r = b_r = A + B_r e_r$, where A and B_r are complex constants. Moreover, let $s = 1$ and $t = 2$, $h_0 = -h_1 = +1$. It will be seen that values which are not infinite can be assigned to the coefficients c_r and c'_r so that equations (2) are satisfied. By substitution in the two series (5) and (6) we obtain the following two series :

$$\left. \begin{aligned} S_1 &\equiv (A + B_0 e_0) + (A + B_1 e_1) + (2A + B_2 e_2 + B_0 e_0) \\ &\quad + (2A + B_3 e_3 + B_1 e_1) + \dots, \\ S_2 &\equiv (A + B_0 e_0) - (2A + B_1 e_1 + B_0 e_0) - (B_2 e_2 - B_1 e_1) \\ &\quad + (2A + B_3 e_3 + B_2 e_2) + (B_4 e_4 - B_3 e_3) - \dots \end{aligned} \right\} \quad (7)$$

It will be seen that both series in (7) are divergent and complex, and that their product is absolutely convergent. Another pair of complex series possessing this property is given at the close of this article.

Since all the terms in the series S_1 are preceded by the positive sign, it is readily seen that any positive integral power of S_1 is a divergent series whose terms increase numerically without limit as r increases without limit. The same conclusion holds for the series S_2 . Since $S_1 \dots S_2 \dots S_1 \dots S_2 \dots$ [to p pairs of factors] $= (S_1 S_2) (S_1 S_2) \dots$ [to p parentheses] $= S_1^p \dots S_2^p$, and since $S_1 \dots S_2$ is an absolutely convergent complex product, it follows that the product of the two complex series S_1^p and S_2^p , the terms of both of which increase numerically without limit as r increases without limit, is absolutely convergent, no matter how large a value the integer p may have.*

If we let a_r and b_r be positive and decreasing monotonously toward zero in such a way that Σa_r and Σb_r are both divergent, if moreover, $t = 2$, $s = 1$, $h_0 = +1$, $h_1 = -1$, then (5) and (6) reduce to the two following series, one divergent, the other conditionally convergent, given by Pringsheim :†

$$\sum_{r=0}^{\infty} (a_{r-2} + a_r) \text{ and } \sum_{r=0}^{\infty} (-1)^{r + \left[\frac{r-1}{2} \right]} (b_{r-1} + (-1)^{r-1} b_r).$$

* See BULLETIN, 2d series, vol. 8 (1902), pp. 233-236.

† See *Transactions of the American Mathematical Society*, vol. 2, p. 408, equation (B). The notation $\left[\frac{r-1}{2} \right]$ signifies here the largest integer contained in $\frac{r-1}{2}$.

In order to deduce from (5) and (6) the pair of conditionally convergent series, whose product is absolutely convergent, which we gave in the BULLETIN, volume 8 (1902), page 231, let $s = t = 4$; $h_0 = h_1 = h_2 = h_3 = +1$; $r = 4v, 4v + 1, 4v + 2$, or $4v + 3$; $r' = 4v$. Let moreover the parenthesis $(a_{r-t} + a_r)$, which we represent for convenience by a'_r , be a real number which is positive when $r = 4v$ or $4v + 1$ and negative when $r = 4v + 2$ or $4v + 3$, and such that $|a'_r| \leq r^{-u}$, where $\frac{1}{2} < u \leq 1$, and $\sum |a'_r|$ is divergent. Without violating conditions (2) we may assume further

$$\begin{aligned} a'_{r'} &= a'_{r'+1}, & |a'_{r'+2}| - a'_{r'} &= e_{r'} \pm w_{r'}, \\ a'_{r'+2} &= a'_{r'+3}, & |a'_{r'+2}| - a'_{r'+4} &= e_{r'+4} \pm w_{r'+4}, \end{aligned}$$

where w_r is a quantity numerically not greater than

$$\frac{1}{(2+r)^{2-u} [\log(2+r)]^A}.$$

Similarly we may assume, without violating conditions (2),

$$h_r b_r - h_{r-1} b_{r-1} = b'_r, \quad |b'_r| \leq r^{-u}, \quad \sum_{r=0}^{\infty} |b'_r| \text{ divergent,}$$

b'_r a real number which is positive when $r = 4v$ and $4v + 2$, and negative when $r = 4v + 1$ and $4v + 3$; also

$$\begin{aligned} b'_{r'} &= b'_{r'+2}, & b'_{r'} - |b'_{r'+1}| &= e_{r'} \pm w_{r'}, \\ b'_{r'+1} &= b'_{r'+3}, & b'_{r'+4} - |b'_{r'+1}| &= e_{r'+4} \pm w_{r'+4}. \end{aligned}$$

There result from these assumptions the two conditionally convergent series $\sum_{r=0}^{\infty} a'_r$ and $\sum_{r=0}^{\infty} b'_r$, whose product is an absolutely convergent series, which were previously given by us in the last mentioned article.

§7. If we are given a series

$$\sum_{r=0}^{\infty} (a_{r-t} + a_r),$$

such that $a_{r-1} - a_r = c_r e_r$, it is quite evident that we can always find a mate for it, such that the product of the two shall be absolutely convergent.

Again it is possible to find two series, every term in the product of which, after the $(t+1)$ th term, vanishes identically. Moreover, the two series may be so chosen that the product possesses the additional property of having for its sum any desired finite number N . To bring this about modify conditions (2) thus,

$$a_{r-1} - a_r = cd^r,$$

where d is any number, and, for $r \geq t$,

$$h_r(b_{r-1} - b_r) = 0.$$

Then $\sum_{r=0}^{\infty} u_r$ is still absolutely convergent. Let t be even and

$$b_{t-1} = db_{t-2} = d^2 b_{t-3} = \dots = d^{t-1} b_0,$$

$$h_0 = -h_1 = h_2 = -h_3 = \dots = -h_t = 1.$$

Then we have, for $r \geq t$,

$$u_r = b_0 cd^r - b_0 d \cdot cd^{r-1} + b_0 d^2 \cdot cd^{r-2} - \dots - b_0 d^{t-1} \cdot cd^{r-t+1} \equiv 0.$$

Under the above conditions the two series (5) and (6) have a product, every term of which, after the $(t+1)$ th term, vanishes identically.

To make the sum of the product of the two series (5) and (6) equal to N , place the sum of the first $t+1$ terms in the product equal to N and then determine the values of d which satisfy this condition.

Thus, let $t=2$, $s=1$, $c=+1$, then from $a_{r-1} - a_r = d^2$, for $r \geq 1$, we get the series (5) (the first term of which is assumed to be a_0)

$$\begin{aligned} a_0 + (a_0 - d) + (2a_0 - d - d^2) + (2a_0 - 2d - d^2 - d^3) \\ + (2a_0 - 2d - 2d^2 - d^3 - d^4) + \dots \end{aligned}$$

Assuming $b_0 = 1$, the series (6) becomes

$$\begin{aligned} 1 - (d+1) + (d-1) + (d+1) - (d-1) - (d+1) \\ + (d-1) + \dots \end{aligned}$$

All the terms in the product of these two series, after the second term, vanish. Putting the sum of the first two terms $= N = 0$, we have $a_0 - d(a_0 + 1) = 0$. If we assume $a_0 = -\frac{1}{9}$, then $d = 10$, and the two factor series become

$$U = -\frac{1}{9} - 11\frac{1}{9} - 112\frac{2}{9} - \dots$$

$$U' = 1 - 11 + 9 + 11 - 9 - 11 + 9 + 11 - \dots$$

Since $U \cdot U' = 0$, we have $U^p \cdot U'^p = 0$. As all the terms in U are of the same sign, it is easily seen that U^p is divergent for all positive integral values of p . U' and U'^p are also divergent.

§ 8. If we assume $t = 2$, $s = 1$, $c = -1$, $a_0 = 1$, $b_0 = 1$, $N = 0$, then the condition that the sum of the product of (5) and (6) shall vanish becomes $d^2 + 1 = 0$ and (letting $i = \sqrt{-1}$) the factor series thus obtained are the two complex divergent series

$$1 + (1 + i) + i + 0 + 1 + (i + 1) + i + 0 + \dots$$

$$1 - (1 + i) + (i - 1) + (i + 1) - (i - 1) - (i + 1) + \dots$$

COLORADO COLLEGE, COLORADO SPRINGS,
April 12, 1902.

THREE SETS OF GENERATIONAL RELATIONS DEFINING THE ABSTRACT SIMPLE GROUP OF ORDER 504.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, October 25, 1902.)

1. CONSIDERABLE interest attaches to the simple group of order 504. The existence of this simple group was discovered by Professor Cole.* This was one of the facts that lead Professor Moore † to his investigation of the linear fractional group in the general Galois field, resulting in the discovery of a new doubly infinite system of simple groups.

* "On a certain simple group," Mathematical Papers, Chicago Congress of 1893.

† BULLETIN, December, 1893; Mathematical Papers, Congress of 1893.

The first set of generational relations follows readily from a general theorem quoted, with references, in § 2. From it is derived, in §§ 5-9, the second set of generational relations

$$A^7 = I, \quad B^2 = I, \quad (AB)^3 = I, \quad (A^3BA^5BA^3B)^2 = I.$$

The same method was followed by the writer * to determine the well-known generational relations for the simple linear fractional groups in the $GF[5]$ and $GF[2^3]$, each of order 60; and for that in the $GF[7]$, of order 168. In a "Note on the simple group of order 504," Burnside † established the above second set by a direct analysis. Other derivations of this second set are due to R. Fricke ‡ and to De Séguier.§

The simplest of the three is the (new) third set

$$C^9 = I, \quad D^2 = I, \quad (CD)^3 = I, \quad (C^3DC^5D)^2 = I.$$

First Set of Generational Relations, §§ 2-4.

2. THEOREM.|| *The group of all linear fractional substitutions of determinant unity in the $GF[p^n]$ is simply isomorphic with the abstract group generated by the operators T and S_λ (λ running through the series of p^n marks of the field), subject to the generational relations*

$$(1) \quad T^2 = I, \quad S_0 = I, \quad S_\lambda S_\mu = S_{\lambda + \mu} \quad (\lambda, \mu \text{ any marks}),$$

$$(2) \quad S_\lambda TS_\mu TS_{\frac{\lambda-1}{\lambda\mu-1}} TS_{-(\lambda\mu-1)} TS_{\frac{\mu-1}{\lambda\mu-1}} T = I$$

(λ and μ any marks such that $\lambda\mu \neq 1$).

3. For $p^n = 2^3$, the group is a simple group of order 504. Let the $GF[2^3]$ be defined by the irreducible congruence

$$i^3 \equiv i + 1 \pmod{2},$$

so that the marks of the field are

$$0, \quad 1, \quad i, \quad i + 1, \quad i^2, \quad i^2 + 1, \quad i^2 + i, \quad i^2 + i + 1.$$

* Linear Groups, pp. 302-303.

† Math. Annalen, vol. 52 (1899), pp. 174-176.

‡ Ibid., p. 335.

§ Journal de Math., series 5, vol. 8 (1902), p. 287.

|| Found, but not yet published, by Professor Moore. A different proof for finite fields is due to the writer: Linear Groups, pp. 300-302.

The relations $S_\lambda S_\mu = S_{\lambda+\mu}$ all follow from

$$(3) \quad S_1^2 = S_i^2 = S_{i+1}^2 = S_{i^2}^2 = S_{i^2+1}^2 = S_{i^2+i}^2 = S_{i^2+i+1}^2 = I,$$

$$(4) \quad S_1 S_i = S_{i+1}, \quad S_1 S_{i^2} = S_{i^2+1}, \quad S_1 S_{i^2+i} = S_{i^2+i+1}, \quad S_{i^2} S_i = S_{i^2+i}.$$

Taking the inverse of every member of (4) and applying (3), we get

$$S_i S_1 = S_{i+1}, \quad S_{i^2} S_1 = S_{i^2+1}, \quad S_{i^2+i} S_1 = S_{i^2+i+1}, \quad S_i S_{i^2} = S_{i^2+i}.$$

To show that $S_i S_\mu = S_{i+\mu}$ for every mark μ , we note that

$$S_i S_{i+1} = S_i \cdot S_i S_1 = S_1, \quad S_i S_{i^2+1} = S_i \cdot S_{i^2} S_1 = S_{i^2+i} S_1 = S_{i^2+i+1}$$

$$S_i S_{i^2+i} = S_i \cdot S_i S_{i^2} = S_{i^2}, \quad S_i S_{i^2+i+1} = S_i \cdot S_{i^2+i} S_1 = S_{i^2} S_1 = S_{i^2+1}$$

Then $S_{i+1} S_\mu = S_{i+1+\mu}$ follows at once, since

$$S_{i+1} S_\mu = S_i S_i S_\mu = S_i S_{i+\mu} = S_{i+1+\mu}.$$

That $S_\lambda S_\mu = S_{\lambda+\mu}$ ($\lambda = i^2, i^2+1, i^2+i, \text{ or } i^2+i+1$) follow quite similarly.

4. For $\lambda = 0$ or 1 or for $\mu = 0$ or 1, relation (2) reduces to *

$$(5) \quad (TS_i)^3 = I.$$

Among the remaining relations (2) occur the following six :

$$(6) \quad S_i TS_i TS_{i^2+i} TS_{i^2+1} TS_{i^2+i} T = I,$$

$$(7) \quad S_{i^2} TS_{i^2} TS_i TS_{i^2+i+1} TS_i T = I,$$

$$(8) \quad S_{i+1} TS_{i+1} TS_{i^2+1} TS_{i^2} TS_{i^2+1} T = I,$$

$$(9) \quad S_{i^2+1} TS_{i^2+1} TS_{i^2+i+1} TS_{i^2+i} TS_{i^2+i+1} T = I,$$

$$(10) \quad S_{i^2+i} TS_{i^2+i} TS_{i^2} TS_{i+1} TS_{i^2} T = I,$$

$$(11) \quad S_{i^2+i+1} TS_{i^2+i+1} TS_{i+1} TS_i TS_{i+1} T = I.$$

From these follow all the relations (2) in which neither λ nor μ is 0 or 1. For, in (6)–(11), each subscript $i, i+1, i^2, i^2+1, i^2+i, i^2+i+1$ is followed immediately by all of the others, excepting its reciprocal, provided the first subscript be regarded

* Thus, for $\lambda = 0$, (2) becomes $TS_\mu TS_1 TS_1 TS_{-\mu+1} T = I$. Transforming both members by TS_μ , we get $TS_1 TS_1 TS_1 = I$.

as following the last; while from (2) results an equivalent relation by using for the initial λ, μ any two consecutive subscripts. We have therefore the following

THEOREM.* *The abstract simple group of order 504 is generated by the operators $T, S_i, S_j, S_{i+1}, S_j, S_{j+1}, S_{i+j}, S_{i+j+1}$ subject to the generational relations $T^2 = I$ and (3), (4), (5), (6), (7), (8), (9), (10), (11).*

Second Set of Generational Relations, §§ 5-9.

5. Transforming both members of (8) by TS_{i+1} , we get

$$(12) \quad S_{i+1}TS_{i+1}TS_{i+1}TS_{i+1} = S_{i+i}TS_{i+i}TS_{i+i}.$$

Denoting each product by P , we form PTP and get

$$S_{i+1}(TS_{i+1})^7 = S_{i+i}TS_{i+i}TS_{i+i}TS_{i+i}TS_{i+i}TS_{i+i}.$$

In view of (10), the second member equals

$$S_{i+i}TS_{i+i}T \cdot TS_{i+i}TS_{i+i} \cdot S_{i+i} = S_{i+i}.$$

Hence

$$(13) \quad (TS_{i+1})^7 = I.$$

Set $A \equiv TS_{i+1}$, $B \equiv S_i$. By (3), (4), (5), (13),

$$(14) \quad A^7 = I, B^2 = I, (AB)^3 = I.$$

6. From (10) and (7), we obtain respectively

$$S_iTS_i = S_{i+1}TS_{i+1}TS_iTS_{i+1}TS_{i+1} = A^{-2}BA^2, \\ S_{i+i+1} = TS_iTS_iTS_iTS_iT.$$

* Note added November 10. The relations (8), (9), (10) may be omitted. Indeed, we may write (6) thus:

$$S_iTS_iTS_{i+i+1}S_iTS_{i+i+1}TS_{i+i+1}T = I.$$

Replacing TS_iTS_{i+i+1} by $S_{i+i}TS_{i+i}TS_iT$, as may be done by (7), and then TS_iT by S_iTS_i , as may be done by (5), we obtain (10), with its subscripts permuted cyclically. Again, we may write (6) thus:

$$S_iTS_{i+1}S_iTS_{i+i}TS_{i+i+1}TS_{i+i+1}T = I.$$

Replacing S_iTS_{i+1} by $TS_{i+1}TS_{i+i+1}TS_{i+i+1}T$, in virtue of (11), and then TS_iT by S_iTS_i , in virtue of (5), we obtain a relation immediately equivalent to (9). Finally, we may write (7) thus:

$$S_{i+i}TS_{i+i}TS_i \cdot S_{i+1}TS_{i+i+1}TS_iT = I.$$

Replacing $S_{i+1}TS_{i+i+1}T$ by $TS_iTS_{i+1}TS_{i+i+1}$, in virtue of (11), and then TS_iT by S_iTS_i , we obtain (8).

But $ABA^{-1} = TS_i T$. Hence

$$(15) \quad S_{i+i+1} = ABA^{-3}BA^3BA^{-1},$$

$$(16) \quad S_{i+1} = B \cdot ABA^{-3}BA^3BA^{-1} = A^{-1}BA^3BA^3BA^{-1},$$

since $BAB = A^{-1}BA^{-1}$ by (14). Since S_{i+1} is of period 2,

$$(17) \quad (A^3BA^3BA^3B)^2 = I.$$

7. We next express all the generators of the group in terms of A and B . Transforming (8) by TS_{i+1} , we get

$$S_{i+1}A^2TS_iS_iTS_iT = I.$$

Replacing S_{i+1} and S_iTS_i by their values and TS_i by AB ,

$$(18') \quad T = A^{-1}BA^3BA^3BA^2BA^5BA^2.$$

Since $BA^2B = A^{-1}BA^{-2}BA^{-1}$ by (14), we get

$$\begin{aligned} T &= A^{-1}BA^3BA^2BA^{-2}BA^4BA^2 \\ &= A^{-1}BA^2BA^{-2}BA^{-3}BA^4BA^2. \end{aligned}$$

Replacing $BA^{-3}BA^4BA^2$, of period 2 by (17), by its inverse, we get

$$(18) \quad T = A^{-1}BA^2BA^3BA^3BA^3B.$$

8. The S_μ are given by (15), (16) and the relations

$$S_1 = TAB, S_{i+1} = T^{-1}A, S_{i+i} = TABS_{i+i+1}, S_i = S_{i+i}B.$$

Some of these relations may be simplified. Thus,

$$\begin{aligned} S_{i+i} &= A^{-1}BA^2BA^3BA^3BA^3B \cdot AB \cdot ABA^{-3}BA^3BA^{-1} \\ &= A^{-1}BA^2BA^3BA^3BA^{-1}BA^3BA^{-1} \quad [\text{by (14)}] \\ &= A^{-1}BA^2BA^3BA^4BA^4BA^{-1} \quad [BA^{-1}B = ABA] \\ &= A^{-2}BA^{-2}BA^2BA^4BA^4BA^{-1} \\ &\quad [BA^2B = A^{-1}BA^{-2}BA^{-1}] \\ &= A^{-2}BA^{-2}B \cdot BA^3BA^3BA^5 \cdot A^{-1} \quad [\text{by (17)}] \\ &= A^{-2}BABA^3BA^4 \\ &= A^{-3}BA^2BA^{-3} \quad [BAB = A^{-1}BA^{-1}]. \end{aligned}$$

Taking the inverse for $S_{\sigma+i}$, we have the relations

$$(19) \quad \begin{cases} S_i = B, S_1 = TAB, & S_{\sigma+i} = A^3BA^5BA^3, \\ S_{i+1} = T^{-1}A, & S_{\sigma} = A^3BA^5BA^3B. \end{cases}$$

9. THEOREM. *The simple group of order 504 is generated by two operators A and B subject to the generational relations (14) and (17).*

It remains to show that the operators defined by (15), (16), (18) and (19) satisfy the relations $T^2 = I$, (3)–(11), in view of relations (14) and (17) alone. We derive at once the relations

$$\begin{aligned} S_i^2 &= I, S_{\sigma+i+1}^2 = I, S_{\sigma+i}^2 = I, S_{\sigma}^2 = I, S_{\sigma}S_i = S_{\sigma+i}, \\ S_{\sigma+i}^2 &= A^3BA^5 \cdot BA^{-1}B \cdot A^5BA^3 \\ &= A^3BA^{-1}BA^{-1}BA^3 = A^7 = I. \end{aligned}$$

Since $BA^2B = A^{-1}BA^{-2}BA^{-1}$ by (14), we get

$$\begin{aligned} T^2 &= (A^5BA^5BA^2BA^3BA^3B)^2 \\ &= A^5BA^5BA^2 \cdot A^2BA^4BA^4 \cdot A^5BA^2BA^3BA^3B \\ &= A^5BA^5 \cdot BA^4BA^4BA^2B \cdot A^2BA^3BA^3B \quad [\text{by (17)}] \\ &= A^5BA^5 \cdot A^5BA^3BA^3 \cdot A^2BA^3BA^3B \quad [\text{by (17)}] \\ &= (A^5BA^3BA^3B)^2 = I \quad [\text{by (17)}]. \end{aligned}$$

That $S_{i+1} = T^{-1}A$ has period 2 is shown as follows:

$$\begin{aligned} ATAT &= BA^2BA^3BA^3BA^5BA^3BA^3BA^3B \\ &= BA^2 \cdot A^2BA^4BA^4 \cdot A^3BA^3BA^3B = I \quad [\text{by (17)}]. \end{aligned}$$

We may show that $S_1^2 = I$ by indicating the square of

$$(20) \quad T^{-1}AB = BA^4BA^4BA^5BA^2B$$

and replacing $BA^{-1}B$ by ABA six times in succession. We have therefore derived all the relations (3) and $T^2 = I$. Next,

$$\begin{aligned} S_1S_i &= TAB \cdot B = T^{-1}A = S_{i+1}; \\ S_1 &= TAB = A^{-1}BA^3BA^3BA^2BA^5BA^3B = S_{\sigma+1}S_{\sigma}, \end{aligned}$$

upon employing for T formula (18') which follows from (18), (17), (14). Hence $S_i S_a = S_{a+1}$. Again,

$$S_i S_{a+i} = S_i S_a S_i = S_{a+1} S_i = (S_i S_{a+1})^{-1} = S_{i+1}^{-1} = S_{a+i+1}.$$

Hence relations (4) have all been derived.

It remains to derive relations (5)–(11). From what precedes, we may make use of relations (1). Since $TS_1 = AB$, (5) follows from (14). We next verify that

$$(21) \quad S_a TS_a = A^{-2} BA^2.$$

Employing $S_a = S_i^{-1} = BA^4 BA^2 BA^4$ and the value of T from (18), we obtain as the condition for (21),

$$A^2 BA^4 BA^2 BA^3 BA^2 BA^3 BA^5 BA^2 B = I.$$

Replacing the first two products $BA^2 B$ by $A^{-1} BA^{-2} BA^{-1}$, we get

$$A^2 BA^3 BA^{-2} BABA^{-2} BA^2 BA^5 BA^2 B = I.$$

Replacing BAB by $A^{-1} BA^{-1}$ and then $BA^4 BA^4 BA^2$ by its inverse, in view of (17), we get

$$A^2 BABA^3 BABA^2 B = I,$$

a relation following readily from (14). As in §6, (10) and (7) follow from (21), (15), (16), and

$$(22) \quad ABA^{-1} = AS_{i+1} S_i S_{i+1} A^{-1} = TS_i T.$$

From (18') and (21) follows (8) as in §7. Note that (10) also follows from (8), and $A^7 = I$, as in §5.

Employing (22), relation (6) may be written

$$B \cdot ABA^{-1} \cdot S_{a+i} \cdot TS_i \cdot S_a TS_a \cdot S_i T = I.$$

Applying (14), (19) and (21), the relation becomes

$$A^{-1} BA^{-2} \cdot A^3 BA^5 BA^3 \cdot AB \cdot A^{-2} BA^2 \cdot BT = I.$$

By (14) we may replace $BA^5 B$ and $BA^{-2} B$ by $ABA^2 BA$:

$$A^{-1} BA \cdot ABA^2 BA \cdot A^4 \cdot ABA^2 BA \cdot A^2 BT = I.$$

Replacing $BA^5 B$ by ABA , we get, by (18),

$$A^{-1} BA^2 BA^3 BA^3 BA^3 BT = T^2 = I.$$

Transforming (11) by T , we get

$$TS_{i+1} \cdot S_{\sigma} TS_{\sigma} \cdot S_{i+1} T \cdot S_{i+1} T \cdot S_i \cdot TS_{i+1} = I.$$

Setting $TS_{i+1} = A$ and $S_i = B$, and applying (21), it becomes an identity.

To verify (9), we write it in the form

$$S_{\sigma+1} \cdot TS_1 \cdot S_{\sigma} TS_{\sigma} \cdot S_{i+1} T \cdot S_i \cdot S_{\sigma} TS_{\sigma} \cdot S_{i+1} T = I.$$

Applying (16), (21), $TS_1 = AB$, $S_{i+1} T = A^{-1}$, $S_i = B$, it becomes

$$A^{-1}BA^3BABABA^{-2}BA = I,$$

which is an identity since $BABAB = A^{-1}$ by (14).

Third Set of Generational Relations, §§10-12.

10. Consider the two operators

$$C = S_i T, \quad D = S_{i+1}.$$

From (6) and (8) we obtain respectively

$$(S_i T)^4 = S_{\sigma} TS_{\sigma+1} TS_{\sigma} T, \quad TS_{\sigma+1} TS_{\sigma} T = S_{i+1} TS_{i+1} TS_{\sigma+1}.$$

Hence, applying also (11), we derive

$$(S_i T)^9 = (S_{\sigma+i+1} TS_{i+1} TS_{\sigma+1})^2 S_i T = I.$$

Also, $DC \equiv S_1 T$ is of period 3 by (5). Hence

$$(23) \quad D^2 = I, \quad C^9 = I, \quad (DC)^3 = I.$$

11. Noting that $C^{-1}DC = TS_{i+1}T$, we may write (11) thus:

$$S_{\sigma+i+1} TS_{\sigma+i} \cdot DC \cdot D \cdot C^{-1} \cdot C^{-1}DC = I,$$

$$S_{\sigma+i+1} TS_{\sigma+i} = C^{-1}DC^2DC^{-1}D = C^{-1}DC^3DC.$$

Multiplying the latter on the left by S_{i+1} and on the right by $S_i T$, we get

$$S_{\sigma} TS_{\sigma} T = DC^{-1}DC^3DC^2 = CDC^4DC^2.$$

Applying this result in (7), we get

$$S_{\sigma+i+1} T = C^6 DC^5 DC^7.$$

We now obtain the formulæ *

* It may be shown that $S_{\sigma+i+1} = DC^7 DC^3 DC^7$.

$$(24) \quad S_{i+i} = S_{i+i+1} T \cdot TS_1 = C^6 DC^5 DC^4 D,$$

$$(25) \quad S_{i+1} = S_{i+i} S_{i+1} = C^6 DC^5 DC^6,$$

$$(26) \quad S_i T = S_{i+i}^{-1} S_i T = DC^3 DC^4 DC^4.$$

Since S_{i+1} is of period 2, it follows from (25) that

$$(27) \quad (DC^3 DC^3)^2 = I.$$

We may write (6) in the form

$$C^2 \cdot S_i T \cdot TS_i \cdot TS_i \cdot S_1 T \cdot S_{i+i} \cdot T = I.$$

Applying (26) twice and (24) once, this relation gives

$$C^2 DC^3 DC^4 DC^{-1} DC^5 DC^4 DC^5 DC^6 D = T.$$

Replacing $DC^{-1}D$ by CDC in view of (23) and then $C^3 DC^5 D$ by $DC^4 DC^6$, in view of (27), we get

$$(28) \quad T = C^6 DC^3 DC^4 DC^5 DC^6 D.$$

From the definitions of A and B at the end of § 5, we get

$$(29) \quad A = TD = C^6 DC^3 DC^4 DC^5 DC^6,$$

$$(30) \quad B = CT^{-1} = CDC^3 DC^4 DC^5 DC^6 DC^3.$$

12. THEOREM. *The simple group of order 504 is generated by two operators C and D , subject to the generational relations (23) and (28).*

In view of the theorem of § 9, it suffices to show that, if A and B are defined by (29) and (30), relations (14) and (17) follow from relations (23) and (27). First,

$$(31) \quad BA = CT^{-1} \cdot TD = CD,$$

so that BA is of period 3 in view of (23). To show that $B^2 = I$, we replace $DC^6 DC^4$ by its inverse $C^5 DC^3 D$ in the indicated square of B and get

$$B^2 = CDC^3 DC^4 \cdot DCD \cdot C^6 DC^4 DC^5 DC^6 DC^3.$$

Replacing DCD by $C^{-1}DC^{-1}$ and, in the result, $DC^3 DC^5$ by $C^4 DC^6 D$, we get

$$B^2 = CDC^7 DCDC^5 DC^6 DC^3.$$

Making similar replacements, we get $B^2 = C^6 DC^9 DC^3 = I$.

In the indicated product AA , we replace DC^3DC^3 by C^6DC^4D , then DCD by $C^{-1}DC^{-1}$, then C^6DC^4D by DC^6DC^3 , then $DC^{-1}D$ by CDC twice, and get

$$(32) \quad A^2 = C^7DC^8.$$

It follows at once that

$$A^6 = C^7DC^6DC^6DC^8.$$

It will equal the expression for A^{-1} given by (29) if

$$C^4DC^6DC^6DC^8 = DC^4DC^5DC^6D.$$

Replacing C^4DC^6D by DC^3DC^5 , the condition becomes

$$I = C^4DC^7DCDC^5DC^6D.$$

Replacing DCD by $C^{-1}DC$ and then C^6DC^4D by DC^6DC^3 , the condition reduces to an identity in view of $D^2 = I$, $C^9 = I$.

Finally, to derive (17), we note that

$$BA^3BA^5BA^3 = BA \cdot A^2 \cdot BA \cdot A^4 \cdot BA \cdot A^2,$$

and substitute the values of BA and A^2 given by (31) and (32). The result $CDC^5DC^4DC^8$ is evidently of period 2 by (23).

Generators of the Linear Fractional Group of Order 504.

13. For the first set of generational relations, we may take

$$T: z' = \frac{1}{z}; \quad S_\mu: z' = z + \mu.$$

Hence, for the second set, we may take

$$A: z' = \frac{1}{z} + i + 1; \quad B: z' = z + i.$$

For this second set, Burnside employs

$$A': z' = iz; \quad B': z' = \frac{z + i^2}{i^2z + 1}.$$

We readily obtain the relations

$$A' = TS_{i+1}TS_iTS_iTS_{i+1} = S_{i^2+i+1}TS_{i^2+i}TS_{i+1}TS_{i+1},$$

$$B' = S_{i+1}TS_{i+1}TS_{i^2+i+1}TS_{i^2+i}.$$

For the third set, we may evidently take

$$C: z' = 1 + (z + i); \quad D: z' = z + i + 1.$$

Note that $S_{i+1}T$ is of period 7; while

$$BA^3: z' = \frac{z + i + 1}{z}$$

is of period 9; also, as a check on (27), that

$$DC^3DC^3: z' = \frac{z + i^3}{iz + 1}.$$

THE UNIVERSITY OF CHICAGO,
October 19, 1902.

GENERATIONAL RELATIONS DEFINING THE ABSTRACT SIMPLE GROUP OF ORDER 660.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, October 25, 1902.)

THE abstract group of order 660, simply isomorphic to the group of linear fractional substitutions of determinant unity taken modulo 11, may be generated by two operators S and T subject to the following relations: *

$$(1) \quad S^{11} = I, \quad T^2 = I, \quad (ST)^3 = I,$$

$$(3 \ 3 \ 3 \ 3 \ 3), \quad (7 \ 7 \ 7 \ 7 \ 7), \quad (2 \ 2 \ 4 \ 8 \ 4), \quad (4 \ 4 \ 9 \ 7 \ 9),$$

$$(5 \ 5 \ 2 \ 9 \ 2), \quad (6 \ 6 \ 8 \ 9 \ 8), \quad (8 \ 8 \ 5 \ 3 \ 5), \quad (9 \ 9 \ 10 \ 8 \ 10),$$

$$(2 \ 3 \ 9 \ 6 \ 7), \quad (2 \ 7 \ 6 \ 9 \ 3), \quad (2 \ 8 \ 3 \ 7 \ 10), \quad (2 \ 10 \ 7 \ 3 \ 8),$$

$$(3 \ 6 \ 4 \ 5 \ 10), \quad (3 \ 10 \ 5 \ 4 \ 6), \quad (4 \ 7 \ 5 \ 6 \ 10), \quad (4 \ 10 \ 6 \ 5 \ 7),$$

where the symbol $(a \ b \ c \ d \ e)$ denotes the relation

$$S^a T S^b T S^c T S^d T S^e T = I.$$

These relations are very redundant. We proceed to reduce

* For a proof of the general theorem due to Moore, see Dickson, Linear Groups, § 278, Corollary. When the field is of order a prime p , we may set $S_i = S$, $S_i = S^i$. By § 279, we need retain only the 16 relations (a, b, c, d, e) given above, since the others follow by a cyclic permutation of a, b, c, d, e . It may be shown that $S^2T, S^3T, S^4T, S^5T, S^6T, S^7T, S^8T, S^9T$ have the respective periods 12, 5, 5, 6, 6, 5, 5, 12, results not used here.

the nineteen to five simple relations. For use in this reduction we note that from (1) follows

$$(2) \quad TST = S^{-1}TS^{-1}, \quad TS^{-1}T = STS,$$

$$(3) \quad TS^2T = S^{-1}TS^{-2}TS^{-1}, \quad TS^{-2}T = STS^2TS.$$

Applying to (2 2 4 8 4) relation (3₁), we get

$$STS^{-2}TS^3TS^8TS^4T = I.$$

$$S^3TS^8T = TS^2TS^{-1}TS^7 = TS^3TS^8,$$

in view of (2₂). Hence (2 2 4 8 4) may be replaced by

$$(4) \quad (S^3TS^8T)^2 = I.$$

By (2 2 4 8 4) we have

$$S^7TS^9T = TS^8TS^4TS^2.$$

Substituting this value in (4 4 9 7 9), we get

$$(5) \quad (S^4TS^6T)^2 = I.$$

To (5 5 2 9 2) we apply (3₁) twice in succession. Transforming the result by S , we obtain (4 4 9 7 9).

¶ In (6 6 8 9 8) we replace S^6TS^6T by $TS^2TS^9TS^2$ given by (5 5 2 9 2). The resulting relation becomes an identity upon applying (2₂) twice.

Transforming (8 8 5 3 5) by S^7 and twice replacing TS^8TS^7 by its inverse S^4TS^6T , in view of (5), we obtain $(ST)^3 = I$.

¶ In (9 9 10 8 10) we apply (2₂) three times in succession and obtain an identity.

Transforming (2 3 9 6 7) by S^5 and replacing S^8TS^3T and TS^7TS^8 by their inverses, given by (4) and (5), we get

$$TS^8TSTSTTS^4T = TS^7(ST)^3S^4T = I.$$

Transforming (2 7 6 9 3) by S^8 and replacing S^5TS^7T and TS^3TS^8 by their inverses, we get

$$TS^4TSTSTTS^8T = TS^3(ST)^3S^8T = I.$$

To (2 8 3 7 10) we apply (2₂) and transform the result by S^{-1} . There results relation (4).

Applying (2₂) to (2 10 7 3 8), we get (4).

In (3 6 4 5 10) we replace TS^6TS^4 by its inverse and get

$$(S^{-1}T)^3 = I.$$

Similarly, (3 10 5 4 6), (4 7 5 6 10), and (4 10 6 5 7) follow by applying first (5) and then (1).

Transforming (3 3 3 3 3) by S^6 and replacing S^3TS^3T by its inverse TS^3TS^3 , given by (4), we get

$$TS^8TS^6TS^3TS^3TS^6 = I,$$

a relation following by taking the inverse of (8 8 5 3 5).

Transforming (7 7 7 7 7) by S^2 and replacing S^5TS^7T by its inverse TS^4TS^6 , given by (5), we get

$$TS^4TS^2TS^7TS^7TS^2 = I,$$

a relation following by taking the inverse of (4 4 9 7 9).

Hence the simple group of order 660 is generated by two operators S and T subject to relations (1), (4), and (5).

THE UNIVERSITY OF CHICAGO,
October 22, 1902.

THE CARLSBAD MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG, SEPTEMBER, 1902.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held at Carlsbad, September 21–26, forming as usual a part of the Versammlung Deutscher Naturforscher und Aertze. At the business meeting of the Society, committees were appointed to make arrangements for the international mathematical congress which will be held at Heidelberg in August, 1904. A committee was also appointed to consider the feasibility of forming at some point in Germany a very complete collection of mathematical literature. Professor Gutzmer resigned from the position of secretary of the society, and Professor Krazer was elected his successor. The meeting of the Society will be held next year at Cassel.

The following papers were read at the meeting :

Professor CZUBER, Vienna : "A theorem in the theory of errors, and its application."

Professor GRÜNWALD, Prague : "Derivatives with variable index as analytic functions of the latter, and a related functional operation."

Professor HASKELL, California : "The representation of certain resultants as determinants."

Dr. JAHNKE, Berlin : "An elementary theory of the theta functions of one and two arguments."

Dr. JAHNKE, Berlin : "The relation of Steiner to Jacobi."

Professors KLEIN, Göttingen, and MEYER, Königsberg : "The progress on the Encyklopädie der mathematischen Wissenschaften."

Professor KOHN, Vienna : "On a certain sextuple of rays."

Professor KOWALEWSKI, Greifswald : "A report on Lie's theory of transformation groups."

Professor KOWALEWSKI, Greifswald : "On a criterion of du Bois-Reymond for the development in Fourier series."

Professor KOWALEWSKI, Greifswald : "A new characteristic of the projective group of a norm curve."

Dr. LIEBMANN, Leipsic : "On the conformal representation of convex surfaces upon the sphere."

Professor MEYER, Königsberg : "On the spheres inscribed in a tetrahedron."

Professor E. MÜLLER, Vienna : "On the theory of linear systems of curves and surfaces of the second order."

Professor F. MÜLLER, Steglitz : "The abbreviations of the titles of mathematical publications."

Professor SCHUBERT, Hamburg : "Enumerative relations for incidence and coincidence in linear space of higher dimensions."

Dr. STEINITZ, Berlin : "A report on polyhedra."

Dr. VON STERNECK, Vienna : "On an analogy to the additive number theory."

Professor WÄLSCH, Brünn : "On binary analysis."

Professor Czuber developed the relations which hold for a linear form of n independent errors of observation, when the errors follow singly the exponential law, and applied the results to direct observations of equal exactness.

Professor Grünwald spoke on his researches in the theory of the derivatives with variable index, and their development in powers of the latter. The coefficients in the development are found by linear functional operations, and are represented by definite integrals. The application to the solution of certain linear differential equations was mentioned.

Sylvester's dialytic method, applied to finding the resultant of forms of more than two variables, gives a determinant of too high an order — the resultant with a foreign factor. Cayley has found the resultant in the form of a determinant without foreign factor for the case of three ternary quadratic forms, and Hesse for the case of three ternary forms of which two are quadratic and one linear, and of four quaternary forms of which three are quadratic and one linear. These results follow (those of Hesse less directly) from the theorem, that if n homogeneous forms in n variables vanish simultaneously, then the Jacobian vanishes, and if the forms are of the same degree, the first derivatives of the Jacobian vanish also. Professor Haskell adds to this theorem that, if the forms are of the same degree, the second derivatives of the Jacobian are proportional to certain linear functions of the second derivatives of the forms, and the application of this theorem gives the resultant of four or of five quadratic forms, homogeneous in four or five variables respectively, in the form of a determinant without foreign factor.

Dr. Jahnke spoke on Caspary's theory of the theta functions and gave some further developments. In Jacobi's lectures the theory of the theta functions was developed from their definition by series of exponential functions. Caspary used this definition but gave the theory a purely algebraic character without function-theoretical considerations, introducing the orthogonal system with Euler's form for the coefficients in order to express the many algebraic relations between the thetas. For further development of the theory only the algebraic and differential identities between the elements of an orthogonal system are required. The introduction to the elliptic transcendents, as given by Caspary, appears elementary enough for the use of students of physics and technology.

In his second paper Dr. Jahnke read some recently discovered letters of Steiner and Jacobi.

Professors Klein and Meyer spoke on the progress of the *Encyklopädie*. The last part of Volume I containing the index will probably appear before Easter. The chief difficulty in Volume II has been overcome, but further publication will not be immediate. Volume III, on geometry, is divided into three

main parts, and articles from each part will appear in the next few months; in Part I, Enriques, Principles of Geometry; in Part II, Dingeldey, Conic Sections; in Part III (Differential Geometry), Mangold, Space Curves and Surfaces. Volume IV (Mechanics) is progressing satisfactorily. Publication will soon begin on Volume V (Physics), with Bryan's article on Thermodynamics, and also on Volume VI (Geodesy and Geophysics).

Professor Kohn's paper is in abstract as follows: There is a single system of six lines, such that every interchange is produced by a collineation or correlation. These lines may be called an equianharmonic ray sextuple from the fact that the four points of intersection of every line that meets four of them are equianharmonic. If an icosahedron form be interpreted on a cubic space curve the six chords given by the six "pairs" of the icosahedron form are such a system; the six corresponding axes form a second system which is related to the first in many ways. Interesting results follow from the fact that two surfaces of second order, one containing three of the lines of the system, the other the remaining three, cut each other in the edges of a skew four-edge. Corresponding to the ten cases where the six lines reduce to two sets of three, are ten tetrahedra, each of which is inscribed in and circumscribed about every other. The forty vertices and forty sides form a Witting configuration, and it follows that the Klein group of 6! collineations is a subgroup of the hyperelliptic collineation group under which the Witting configuration is invariant. The above is in close relation to results of Klein, Witting, Maschke, and Burkhardt.

Professor Kowalewski's first paper was a report on Lie's theory of transformation groups. Much of Lie's work has a value apart from his group theory, *e. g.*, the idea of element and element union, important in the theory of differential equations; that of contact transformations, which Lie conceived in its generality in the first years of his activity; and, of greatest importance, the infinitesimal transformation. Lie was in possession of most of these ideas in 1873 when he began to develop his theory of transformation groups, above all, however, that of the infinitesimal transformation. He had found in 1872 that an ordinary differential equation of first order with a known

infinitesimal transformation could be integrated by quadrature, and had proposed to himself the problem: Of what value for the integration of a system of differential equations is the knowledge of infinitesimal transformations which leave the system unaltered? He considered the case of a complete system of linear partial differential equations in detail. If two infinitesimal transformations Xf and Yf leave such a system unchanged, then this holds also for the "Klammerausdruck"

$$X(Yf) - Y(Xf) = (XY)f.$$

By this theorem the question is reduced to the consideration of a complete system, with r known infinitesimal transformations $X_1f, \dots, X_rf$, which satisfy relations of the form

$$(X_iX_k)f = \sum C_{iks}X_sf,$$

and these relations hold for all point transformations. This was the starting point for the theory of transformation groups. Lie found by involved reckoning that r independent infinitesimal transformations which satisfy relations of the above form, belong to a continuous group with r parameters, and showed conversely that every such group contains r independent infinitesimal transformations which satisfy the above relations. He had then, although without rigorous proofs, the two fundamental theorems of his group theory. The first presentation of Lie's results is in an article by Klein and Lie in the *Göttinger Nachrichten*, December 3, 1874. The differential equations of his later first fundamental theorem occur here, while the third, which relates to the constants C_{iks} , appeared in 1876 in the *Norwegian Archiv*. In the Göttingen note Lie indicates some applications of his theory to ordinary differential equations, which he did not complete until 1882 (*Archiv for Mathematik*, volumes 8 and 9). In 1882 Lie applied his theory of finite groups to infinite groups by considering only such groups as can be defined by differential equations. He showed also that every such group determines infinitely many differential invariants (*Mathematische Annalen*, volume 24).

A norm curve in R_n , i. e., a rational curve of order n which is contained in no plane point manifold of R_n , admits a group of ∞^3 projective transformations. Professor Kowalewski, in his second paper, gave a complete enumeration of all projective transformation groups which contain the above as a subgroup.

For $n > 2$ there exists in general (except for the group of all projective transformations) only one such group. The single exception is in case of R_6 , where a fourteen group exists, which has the Killing constitution and plays a part in the work of Engel.

In his third paper, Professor Kowalewski calls attention to a little known criterion for the convergence of a Fourier series which was published by P. du Bois-Reymond in 1881 in the *Comptes Rendus* and earlier in *Crelle*. A simpler proof of this criterion was indicated.

Closely related to the theorem that a closed convex surface can not in general be deformed (verbogen), without introducing singularities, is the following theorem: A closed convex surface can not in general be conformally represented, without singularities, upon the sphere. Dr. Liebmann considered a special surface of revolution which admits at most a three parameter group of regular conformal transformations, while for the sphere the six parameter group of homographic transformations forms such a regular conformal group. The surface cannot, therefore, be represented conformally upon the sphere without singularities, and the theorem holds for this special surface. Other such examples can be found showing that the theorem in general holds.

A theorem of Schwarz states that if on each face of a tetrahedron circumscribing a sphere lines are drawn from the point of tangency to the vertices, then the corresponding angles about the points of tangency on the four faces are equal. Interpreting this theorem by projective geometry, Professor Meyer gave some interesting generalizations.

Projective geometry is generalized after Klein by considering the properties of geometrical configurations invariant under a more general transformation group. Of special importance is the projective geometry of curves and surfaces of the second order. A simple theorem of this geometry may contain many results from the standpoint of the ordinary projective geometry. Professor E. Müller showed this by applying the theorem, that in a web of conics (Kegelschnittgewebe) two families always have a common conic, to two special webs,

first, one with a double point, and second, one whose curves are tangent to two lines. Three conics ϕ_1, ϕ_2, ϕ_3 , of which ϕ_3 has double tangency with ϕ_1 , determine a web of the first kind, and from the theorem follows that in the family (ϕ_3, ϕ_2) there are three conics each of which passes through a pair of opposite vertices of the tangential complete quadrilateral common to ϕ_1 and ϕ_2 . The theorem of Darboux follows when ϕ_3 is a point pair, and theorems of Chasles on confocal conics result through metric specialization. The application of the above general theorem to the second web also gives interesting theorems, *e. g.*, the point pairs of two projective point rows of the plane belong to a web of this kind. The fruitfulness of similar considerations for higher dimensions was indicated.

Professor F. Müller spoke on the abbreviations of the titles of mathematical journals, and presented a list of some 600 proposed abbreviations. This list will be manifolded for the present, and eventually published.

In the enumerative geometry only those incidence formulæ have been generalized to linear space of n dimensions, by Schubert, Pieri and others, which relate to a line and incident point. Professor Schubert now attacks the general problem, and finds all the fundamental incidence and coincidence formulæ for two linear spaces of dimensions n and $n + m$.

Dr. Steinitz gave a report on the general theory of the polyhedron, starting from the results of Euler. The symbolic table of Catalan, the work of Eberhardt on polyhedron "fields" and the general question of polyhedra of multiple connectivity were treated at some length.

In the additive number theory the number of ways is sought in which a given number can be formed by the addition of given elements. An analogous question is the following: If n and M are given numbers, in how many ways can a number congruent to $n \pmod{M}$ be formed additively from the elements $1, 2, \dots, M-1$. Stern (1863) studied the question for prime moduli. In the paper of Dr. von Sterneek the general problem was considered, with following results: If $(n)_i^0$ is the number of such representations of $n \pmod{M}$ by i elements, and $(n)_i$ the similar number when zero may appear as an element, then

$$(n)_i^0 = (n)_i - (n)_{i-1} + (n)_{i-2} - \cdots + (-1)^i (n)_0,$$

$$i(n)_i = \sum (n - e)_{i-1} - \sum (n - 2e)_{i-2} + \cdots + (-1)^i \sum (n - ie)_0.$$

In the summations e assumes all values of a complete remainder system with respect to M , and $(n)_0 = 0$ except when $n \equiv 0 \pmod{M}$, in which case $(n)_0 = 1$.

The difference between the numbers of even and odd representations of n is then

$$\Delta(n) = \sum (-1)^i (n)_i^0.$$

In particular, for $M \equiv p^k$,

$$\Delta(n) = 0, \quad -p^{k-i}, \quad p^k - p^{k-i}$$

according as n is not divisible by p^{k-i} , is divisible by p^{k-i} , or is divisible by p^k .

For any $M = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ the theorem holds: If n contains any p_i to a power $\equiv a_i - 2$, then $\Delta(n) = 0$. If n contains the factors $p_1, p_2, \dots, p_i$ to the powers $a_1 - 1, a_2 - 1, \dots, a_i - 1$, and the factors $p_{i+1}, \dots, p_k$ to the powers $a_{i+1}, \dots, a_k$, then

$$\Delta(n) = (-1)^i \frac{\phi(M)}{(p_1 - 1)(p_2 - 1) \cdots (p_i - 1)}.$$

If n contains all the factors of M to the same degree, then $\Delta(n) = \phi(M)$. $\phi(M)$ is the Euler function.

Further relations can be developed from the theorem: The difference $\psi(M)$ between the number of odd and even representations when the H. C. F. of the elements is relative prime to the modulus, satisfies the equation $\sum \psi(d) = \phi(M)$, where d takes on the values of all factors of M , and is then the number theory derivative of the Euler function $\phi(M)$.

Professor Wälsch spoke on the application of the theory of binary forms to geometry and mechanics. By the introduction of the Kugelkreis, points, lines, and planes are represented by binary forms. A point a lies in a plane b when the second "Ueberschiebung" of the form a with b is unity. A vector can also be represented by a binary form, and the first and second "Ueberschiebungen" of two forms give the two vector products. A rational space curve of order n is given by the form $r_v^2 s_v^2 / c_v^n$. In applying the binary form to mechanics, the

expressions for the ellipse of inertia, kinetic energy, and the equations of motion, take very simple forms.

I wish to thank the speakers for the use of manuscripts, and to express my indebtedness to many who gave me abstracts of their papers.

C. M. MASON.

GÖTTINGEN,
October, 1902.

SHORTER NOTICES.

Compte rendu du deuxième Congrès international des Mathématiciens tenu à Paris du 6 au 12 août 1900. Procès-verbaux et Communications publiés par E. DUPORCQ. Paris, Gauthier-Villars, 1902. 8vo. 455 pp.

THE members of the second international congress of mathematicians may have felt a little annoyance at the delay in the publication of the official report; but now that this is at last in their hands, the delay no longer appears unreasonable. A glance through the volume makes it plain that the editor's task has been very serious, inasmuch as its execution has depended on the good pleasure of so many contributors, and these so widely dispersed, that it has been quite beyond his power to expedite matters. Editor and publisher are to be congratulated on the beautiful volume which has recently appeared; it was worth waiting for.

The first 26 pages contain the minutes of the sectional and general meetings; pages 27-153 are occupied by the four principal addresses, with which is placed also a French translation of Hilbert's survey of the future problems of mathematics; pages 154-450 are devoted to the sectional papers. Of these thirty-two are given, thus leaving six or eight unreported for reasons beyond the editor's control. In general the papers appear in a French version, the exceptions being one in Italian and eight in English. As a matter of fact, the majority of the communications were made in French, but there were certainly some in German.

The most interesting articles for present reading are naturally those of a somewhat general character, for the results of more special interest have not been held back pending this publica-

tion. The principal addresses have already been given in outline in the BULLETIN,* with the exception of Mittag-Leffler's account of the correspondence between Weierstrass and S. Kowalevski. This is perhaps as interesting as anything in the book. At the end of the volume there is given, in a French translation, a paper by Veronese, "Les postulats de la géométrie dans l'enseignement," which was intended for the Congress, though it was not actually presented then. Among the shorter articles the most satisfying and appropriate are those of the character of an aperçu, as for instance d'Ocagne, "Sur les divers modes d'application de la méthode graphique à l'art du calcul"—Padé, "Aperçu sur les développements récents de la théorie des fractions continues"—and Amodeo, "Coup d'œil sur les courbes algébriques au point de vue de la gonalgité." Amodeo, in particular, has given a most interesting résumé of results obtained regarding groups of points, g_k^1 , cut out on a curve of order m by adjoints of order $m - 3 - a$, adjoints whose existence presupposes the condition $p \equiv \frac{1}{2}am + 1$. But different readers will naturally find most to interest them in different papers.

CHARLOTTE ANGAS SCOTT.

Probabilités et Moyennes Géométriques. Par EMANUEL CZUBER. Traduit par HERMANN SCHUERMANS. Préface de CHARLES LAGRANGE. Paris, A. Hermann, 1902. xi + 244 pp.

THE various problems considered in the direct theory of probability may be roughly divided, as regards the number of possible cases, into four classes. First, the ordinary problems arising in connection with games of chance, where the number of cases is limited, the methods employed being those of combinatorial analysis. Second, problems where the number of cases is still finite, but very large, so that recourse must be made to approximate results based upon Stirling's formula. Here belong, for example, Bernoulli's theorem and the so-called law of large numbers. The third class includes those problems in which the number of cases is infinite, depending upon the values of a certain number of arbitrary parameters; while in a fourth class we may place the problems depending upon arbitrary laws or functions. The last class has as yet received little attention. It may be remarked in passing that

* BULLETIN, vol. 7 (1900-1901), pp. 57-79.

there is another type, intermediate between the second and third, in which the number of cases is infinite, but forms merely a denumerable assemblage. Such problems arise naturally in connection with the ordinary theory of numbers, but so far as the reviewer knows have never been considered.

Professor Czuber's well known monograph, now appearing in an attractive French translation, is devoted to the third class of problems, and the related questions of mean value. The term geometric or local probability is justified, as is remarked in the introduction, in view of the fact that if a problem depends upon arbitrary parameters, it is possible to translate it into an equivalent geometric problem relating to points or more complicated configurations taken at random in certain regions. The three chapters on local probability treat respectively 1° Points taken at random on lines or surfaces or in space ; 2° Lines taken at random in a plane or in space ; 3° Planes taken at random. The discussion of mean value, forming the second part of the book, is much briefer, occupying only a single chapter of sixty pages.

The questions considered in this youngest branch of the theory of probability have for their type the famous needle problem of Buffon, proposed and solved by the latter in his *Essai d'arithmétique morale*, published 1777 in the fourth volume of the *Supplément à l'histoire naturelle*, but composed some years earlier. Laplace in his *Théorie*, pages 359-362, treated the same problem, without however referring to Buffon. It may be remarked that the latter proposed also a more complicated problem, where the plane is ruled into congruent rectangles by means of two sets of parallels, but his solution is incorrect. Laplace gave the solution for the case where the needle or rod is shorter than both dimensions of the rectangle. Todhunter in his *History of the mathematical theory of probability*, § 650, considers the more general problem, but his results, according to Czuber, are incomplete. Since Laplace the chief contributions to the subject are due to Barbier, Jordan and Lalanne of the French school, and McColl, Sylvester and Crofton of the English school.

Of especial interest are the accounts, in connection with some of the problems discussed by Professor Czuber, of experiments verifying the theoretical results. Thus in connection with Buffon's problem, Professor Wolf in 1850, using a rod four fifths the distance between the parallels, found that in 5000 trials

there were 2532 cases of intersection. The theoretical probability in this case is $8/5 \pi$, so that the probable value of π , obtained by comparison with the experimental probability, is 3.1596. No mention however is made of a similar series of experiments reported by De Morgan in his *Budget of Paradoxes*, page 172, leading to the still more accurate value 3.1412. It would be of interest if some one would undertake to carry out the experimental rectification of curves suggested on page 116 of Czuber's book, and based upon Crofton's theorem that the number of random lines which intersect a closed convex curve is proportional to the length of the curve.

It is a commonplace to remark that in no branch of mathematics is it so easy to make errors as in the theory of probability. The development of the theory is marked by a continual revision of the results of earlier investigators. Especially is this true of geometrical probability, where the essential difficulty lies in the definition of what is to be regarded as equally possible, or, as Venn happily expresses it, in "the idealization of the randomness." Mathematically this amounts to fixing the independent parameters which are to be regarded as equicrescent in the problem considered. In the present book only one error of any importance has come to the notice of the reviewer. This occurs in finding the probability that three segments, each less than a given limit, shall determine an acute or obtuse triangle respectively (pages 29, 30). The results obtained, $1 - \pi/4 = .216$ and $\pi/4 = .784$, cannot both be correct since there is a third case, not considered in the text, where the three segments do not determine any triangle. The error is not corrected in the translation, so it may be worth while to give here the correct results, which are $1 - \pi/4 = .216$, $\pi/4 - \frac{1}{2} = .285$, $\frac{1}{2} = .500$, respectively, for the three cases mentioned.

The English student finds introductions to the subject of local probability in the excellent chapters in Todhunter's and Williamson's *Integral Calculus* (the latter chapter written by Professor Crofton); but of course these cannot compare in completeness with a monograph. The present translation should therefore prove welcome to the large class of students who find it more economical to read French than German.

E. KASNER.

Annuaire des Mathématiciens, 1901–1902 ; publié sous la direction de MM. C.-A. LAISANT, AD. BUHL. Paris, C. Naud, 1901. 8vo. xxiv + 469 pp.

THE idea of publishing an annual list of those interested in mathematics was brought forward with much earnestness by Professor F. Rudio at the first international congress of mathematicians at Zurich in 1897. The matter was not taken up, however, until two years later, when M. Laisant, recognizing the favorable opportunity afforded by the international congress to be held at Paris, presented the proposition to M. Naud, the publisher. The plan contemplated an alphabetic list of living mathematicians, with their leading titles, the mathematical societies to which they belong, and their addresses. For the purposes of the compilation a mathematician was defined as a person belonging to one of the following groups : (1) Members of mathematical societies, or of mathematical sections of general scientific societies ; (2) authors of mathematical works of an original nature ; (3) those who teach mathematics as their special department of work. As a result of the labors of the compilers, more than six thousand names appear in the register.

The work also contains lists of mathematical societies, of associations having mathematical sections, and of the journals devoted to this science. A brief necrology of mathematicians, covering the years 1900 and 1901, is made the more valuable by an added biography of Hermite. Unfortunately all dates are omitted in the necrology, a fault which will probably be remedied in future editions. The work closes with a half dozen "notices scientifiques," mostly foreign to the subject in hand :—By Paul Appell : "Sur le principe de la moindre contrainte de Gauss"—By J. Petersen : "Les 36 officiers"—By A. G. Greenhill : "Les fonctions elliptiques au point de vue de leurs applications."—By Ch. Méray : "La langue internationale auxiliaire *Espéranto* et la littérature scientifique."—By P. H. Schoute : "Le nombre des points, des droites, des plans, etc., contenus dans un hyperespace linéaire."—By P. H. Schoute : "La *Revue semestrielle des publications mathématiques*."

The general plan of the work is very commendable. Such a publication puts scientific workers more closely in touch with one another, and makes the mathematical world seem more real. It places at the student's hand a mass of information

that it has heretofore been exceedingly difficult to reach, information especially welcome to those having editorial duties to perform.

Unfortunately, however, the list has been so hastily compiled as to admit a number of errors that are, to say the least, unfortunate. Were it not that this is the initial number, that new ground is being broken, and that the task of condensing several thousand replies was so great, reviewers would be justified in seriously criticising the whole work. As it is, they will doubtless rest content with pointing out certain lines for improvement, hoping, as they are probably justified in doing, that the second number will remedy the defects of the first.

To justify the intimation of inaccuracy a few types of the errors contained in the work may be mentioned. From the list it appears that Lord Kelvin lives in Glasgow, England; that there is a Brogham Young College in Utah, and a Harvard University at Cambridge; that Professor Byerly's middle name is Elwood, and that the well-known Münster mathematician is Professor Killing. Several prominent names are omitted entirely, including W. S. Burnside, W. W. R. Ball, A. W. Panton, and E. M. Langley, while, as if to atone for this neglect, the name of Professor P. F. Smith appears twice on the same page, and that of the late Professor Craig appears both in the list of living mathematicians and in the necrology. The names of the American states are frequently omitted, and occasionally the state is given without the city. In the list of imperfections should also be mentioned the omission of the initials of a large number of French mathematicians, a national habit but an international annoyance.

It does not seem too much to hope that the second volume will be critically revised by a good linguist, that an effort will be made to enlarge the list judiciously, that the necrology will contain the dates of birth and death, that the list of periodicals will be made more nearly complete, that the notes will be submitted for correction to the various persons concerned, and that the extraneous matter will give place to topics more germane to the subject in hand.

DAVID EUGENE SMITH.

NOTES.

THE approaching Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY will extend through the two days, Monday and Tuesday, December 29–30. The Council will meet on Monday morning, and the annual election of officers and other members of the Council will be held on Tuesday morning. At the opening of the afternoon session on Monday, the retiring President, Professor ELIAKIM HASTINGS MOORE, will deliver the presidential address, the subject of which will be: "The Foundations of Mathematics."

ON recent academic occasions, the AMERICAN MATHEMATICAL SOCIETY has been represented by delegates as follows: At the Abel Centenary, by Professor E. B. VAN VLECK and Dr. E. B. WILSON; at the installation of President James, of Northwestern University, by President E. H. MOORE; at the installation of President Swain, of Swarthmore College, by Professor CHARLOTTE ANGAS SCOTT. The Society has recently received an invitation to send delegates to the coming centenary of the University of Jurievi (Dorpat).

THE forthcoming report of the librarian of the AMERICAN MATHEMATICAL SOCIETY will show that the library has made satisfactory progress during the year. There are now about twice as many periodicals on the exchange list as there were a year ago, about three times as many bound volumes of a non-periodical character, and over twenty times as many bound volumes of journals. The total number of bound volumes has increased from 121 to over 900. Several noteworthy gifts have been made to the library, including Bowditch's translation of Laplace's *Mécanique céleste*, and two volumes by the late Professor Benjamin Peirce from his own collection. An appeal is again made to members to present to the library copies of their works, whether elementary or advanced, to the end that the collection may become an exponent of the country's mathematical progress. It is also suggested that members may have copies of earlier works, including volumes of journals, which they would be willing to donate for such a purpose. Such gifts would be very welcome, especially if published in this country. Since the Society has no funds with which to make

purchases, it is forced to depend solely upon the generosity of its friends and upon its accessions in the way of exchanges.

THE National academy of sciences held its autumn meeting at Johns Hopkins University on November 11 and 12. No mathematical papers were presented.

At the meeting of the London mathematical society held on November 13, the following papers were read:—By Dr. E. W. HOBSON: "On the infinite and the infinitesimal in mathematical analysis."—By Professor D. HILBERT: "On the proposition concerning the equality of the base angles in an isosceles triangle."—By Professor A. C. DIXON: "Expansion by means of Lamé's functions."—By Mr. W. H. YOUNG: "Note on unclosed sets of points defined as the limits of a sequence of closed sets of points."—By Professor H. LAMB: "Wave propagation in two dimensions."—By Professor M. J. M. HILL: "The continuation of certain fundamental power series."—By Mr. L. CRAWFORD: "A geodesic on a spheroid and an associated ellipse."

AN appeal has been sent out by the treasurer of the Royal society for financial support in completing the author catalogue of scientific papers from 1883 to 1900, and in publishing the subject index for the nineteenth century. The list of papers, arranged by authors, has been completed to 1883, at an expense of nearly \$75,000, and the work on the list arranged by subjects has been partly done. The work of cataloguing papers issued since the beginning of the twentieth century has become so burdensome that the Royal Society has turned it over to an international committee.

THE second international congress of mathematicians will be held at Heidelberg in 1904.

COLLEGE OF FRANCE. — The following mathematical courses are given during the first semester of the present academic year, opening December 1, 1902:—By Professor CAMILLE JORDAN: Differential equations, two hours.—By Professor M. BRILLOUIN: Electric waves, two hours.—By Professor J. HADAMARD: Calculus of variations, one hour; Differential equations of continuous media, one hour.—By Professor LEBESGUES: Definite integrals and Fourier series, one hour.

It is of interest to recall that the chair now held by Professor Lebesgues was established a few years ago by M. Peccot under

the peculiar condition that the appointee should be under thirty years of age. The tenure of office is three years. M. Lebesgue's predecessor was M. E. Borel, whose lectures on Divergent series (1899-1900), Series with positive terms (1900-1901), and Meromorphic functions (1901-1902), recently published by Gauthier-Villars, present substantial evidence of the wisdom of M. Peccot's foundation. Both M. Borel and M. Lebesgue are graduates of the Ecole normale supérieure.

At the Ecole normale supérieure, Professor J. TANNERY lectures on Galois theory and Professor P. Painlevé on Abelian functions.

PROFESSOR J. W. GIBBS, of Yale University, has been elected a corresponding member of the Munich academy of sciences.

MR. R. TUCKER retired from the position of honorary secretary of the London mathematical society on November 13. He has held this office since 1867, having been elected soon after the organization of the society.

A PORTRAIT of the late Professor P. G. Tait was unveiled at Peterhouse, Cambridge University, in October, by Lord Kelvin. Addresses were made by Lord Kelvin and by Sir George Stokes, master of Pembroke.

MR. R. W. H. T. HUDSON, B.A., has been appointed lecturer in mathematics at University College, Liverpool.

DR. H. S. CARSLAW lecturer in mathematics in the University of Glasgow, and fellow of Emmanuel College, Cambridge, has been appointed professor of pure and applied mathematics at the University of Sydney.

PROFESSOR MARTIN DISTELI, of Carlsruhe, has been called to the University of Strassburg as associate professor of mathematics.

DR. K. DÜHLEMANN, of Munich, has been advanced to the associate professorship in descriptive geometry.

PROFESSOR E. MÜLLER, of Königsberg, Prussia, has been called to the Technische Hochschule, at Vienna, as professor of mathematics.

DR. M. RADAKOVIC, of Innsbruck, has been advanced to an associate professorship of mathematics.

PROFESSOR D. KIKUCHI, president of the University of Tokio, recently created a baron by the emperor, has been made minister of education for Japan. Dr. SADAJI TAKAGI, of Tokio, has been advanced to an associate professorship of mathematics.

PROFESSOR F. KOLÁČEK, of Brunn, has been called to Prague as professor of mathematical physics in the Bohemian University.

PROFESSOR W. H. BOUGHTON, of Denison University, has been appointed professor of civil engineering in the University of West Virginia.

THE death is announced of Professor NIKOLAUS BUDAJEW, of St. Petersburg, at the age of sixty-nine years.

GENERAL ANNIBALE FERRERO, director of the Italian geodetic survey, died on August 7, in Rome.

PROFESSOR P. H. PHILBRICK died at Medford, Oregon, October 10, 1902. Dr. J. W. DAVIS died in New York, November 7, 1902. Both were members of the AMERICAN MATHEMATICAL SOCIETY.

THE following catalogues of second-hand mathematical works have recently appeared: R. Friedlaender & Sohn, 11 Carlstrasse, Berlin, N. W., no. 420, general mathematics, 114 pages; no. 421, calculus of probabilities, 16 pages. Mayer and Müller, 2 Prinz Louis Ferdinandstrasse, Berlin, N. W., no. 189, history of mathematics and other sciences, 5295 numbers; A. Herrman, 6 Rue de la Sorbonne, Paris, no. 76, part 2, mathematical works from the library of the late Joseph Bertrand, 88 pages.

PROFESSOR G. W. GREEN, of the department of mathematics in the Illinois Wesleyan University, died at Bloomington, Ill., on December 10, aged forty-five years.

PROFESSOR HENRY MITCHELL, formerly of the Massachusetts Institute of Technology and well known for his work on the U. S. Coast and Geodetic Survey, died in New York on December 1, at the age of seventy-two.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- ABEL (N. H.). Mémorial publié à l'occasion du centenaire de sa naissance. Kristiania [1902]; Leipzig, Teubner. 4to. 13 + 119 + 135 + 61 + 64 + 59 + 2 pp, 2 plates, 6 facs. M. 21.00
- BENOIST (A.). See CLEBSCH (A.).
- BLICHFELDT (H. F.). On the determination of the distance between two points in space of n dimensions. 4to. 1902. (*Transactions of the American Mathematical Society* 3, pp. 467-481.)
- BLUM (R.). Cykloiden und Cykloidalen als Umhüllungskurven und deren Zusammenhang mit den Fusspunktkurven der Kegelschnitte. (Progr.) Stuttgart, 1902. 4to. 56 pp.
- BOHLMANN (G.). See SERRET (J. A.).
- CLEBSCH (A.). Leçons sur la géométrie, recueillies et complétées par F. Lindemann et traduites par A. Benoist. (En 3 volumes.) Vol. I: Traité des sections coniques et introduction à la théorie des formes algébriques. Nouveau tirage. Paris, Gauthier-Villars, 1903. 8vo. Fr. 12.00
- CZUBER (E.). Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung. 1ste Hälfte. Leipzig, Teubner, 1902. 8vo. Pp. 1-304. (Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. IX, 1.) M. 12.00
- DUNKEL (O.). Regular singular points of a system of homogeneous linear differential equations of the first order. 8vo. 1902. (*Proceedings of the American Academy of Arts and Sciences* 38, pp. 341-370.)
- EMCH (A.). Algebraic transformations of a complex variable realized by linkages. 4to. 1902. (*Transactions of the American Mathematical Society* 3, pp. 493-498.)
- ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III, 3. Heft 1: H. von Mangoldt, Anwendung der Differential- und Integralrechnung auf Kurven und Flächen; R. von Lilienthal, Die auf einer Fläche gezogenen Kurven. Leipzig, Teubner, 1902. 8vo. Pp. 1-183.
- ENGEL (F.). See GRASSMANN (H.).
- ESTANAVE (E.). Nomenclature des thèses des sciences mathématiques, soutenues en France dans le courant du 19e siècle devant les Facultés des sciences de Paris et des départements. Paris, Gauthier-Villars, 1902. 8vo. Fr. 2.00
- . Revue décennale des thèses présentées à la Faculté des sciences de Paris, en vue du grade de docteur ès sciences, du 1er janvier 1891 au 31 décembre 1900, avec l'indication des périodiques contenant la plupart des ces mémoires ou leurs analyses. Paris, Gauthier-Villars, 1901. 8vo. 115 pp. Fr. 5.00

- FRICKE (R.). Hauptsätze der Differential- und Integralrechnung, als Leitfaden zum Gebrauch bei Vorlesungen zusammengestellt. 3te, umgearbeitete Auflage. Braunschweig, Vieweg, 1902. 8vo. 15 + 218 pp. M. 5.00
- GOTTSCHALK (A.). Die konforme Abbildung gewisser krummlinig begrenzter Vielecke. II. (Progr.) Münster, 1902. 8vo. 20 pp. M. 1.20
- GRASSMANN (H.). Gesammelte mathematische und physikalische Werke. Auf Veranlassung der mathematisch-physischen Klasse der k. sächsischen Gesellschaft der Wissenschaften herausgegeben von F. Engel. Vol. II. Teil 2: Die Abhandlungen zur Mechanik und zur mathematischen Physik, herausgegeben von J. Lüroth und F. Engel. Leipzig, Teubner, 1902. 8vo. 8 + 266 pp. M. 14.00
- GROSSMAN (M.). Ueber die metrischen Eigenschaften kollinearer Gebilde. Frauenfeld, 1902. 4to. 27 pp. M. 1.50
- HAMBURGER (M.). Gedächtnisrede auf Immanuel Lazarus Fuchs, geboren am 5. V. 1833, gestorben am 26. IV. 1902. Mit dem Bildnis des Verstorbenen, sowie einem Verzeichnis seiner Schriften. Leipzig, Teubner, 1902. 8vo. 16 pp. M. 1.00
- HARNACK (A.). See SERRET (J. A.).
- HAUSSNER (R.). See SCHERING (E.).
- HOEFER (F.). Histoire des mathématiques, depuis leurs origines jusqu'au commencement du 19e siècle. 5e édition. Paris, Hachette, 1902. 16mo. 3 + 609 pp. Fr. 4.00
- HUTCHINSON (J. I.). See SNYDER (V.).
- JAHRBUCH über die Fortschritte der Mathematik, begründet von C. Ohrtmann. Herausgegeben von E. Lampe und G. Wallenberg. Vol. 31: 1900. Heft 2. Berlin, Reimer, 1902. 8vo. Pp. 481-672. M. 6.00
- KEYSER (C. J.). Mathematical productivity in the United States. 8vo. 1902. (*Educational Review* 1902, pp. 346-357.)
- KOWALEWSKI (G.). See PFAFF (J. F.).
- LAMPE (E.). See JAHRBUCH.
- LANDFRIEDT (E.). Theorie der algebraischen Funktionen und ihrer Integrale. Leipzig, Göschen, 1902. 12mo. 4 + 294 pp. Cloth. (Sammlung Schubert, No. XXXI.) M. 8.50
- . Thetafunktionen und hyperelliptische Funktionen. Leipzig, Göschen, 1902. 12mo. 4 + 155 pp. Cloth. (Sammlung Schubert, No. XLVI.) M. 4.50
- LIEBMANN (H.). See LOBATSCHESKIJ (N. I.).
- LILIENTHAL (R. VON). See ENCYKLOPÄDIE.
- LINDEMANN (F.). See CLEBSCH (A.).
- LOBATSCHESKIJ (N. I.). Pangeometrie. (Kasan, 1856.) Uebersetzt und herausgegeben von H. Liebmann. Leipzig, Engelmann, 1902. 12mo. 95 pp. Cloth. (Oswald's Klassiker der exakten Wissenschaften, No. 130.) M. 1.70
- LÜROTH (J.). See GRASSMANN (H.).
- MANGOLDT (H. VON). See ENCYKLOPÄDIE.

NOETHER (H.). See RIEMANN (B.).

PETERSEN (J.). Analytisk plangeometri. 4te udgave. Del I. Kjobenhavn, 1902. 8vo. Pp. 1-96. M. 2.50

PFAFF (J. F.). Allgemeine Methode, partielle Differentialgleichungen zu integrieren. (1815.) Aus dem Lateinischen übersetzt und herausgegeben von G. Kowalewski. Leipzig, Engelmann, 1902. 12mo. 84 pp. Cloth. (Ostwald's Klassiker der exakten Wissenschaften, No. 129.) M. 1.40

PICKLER (A.). Ueber die Auflösung der Gleichung $\phi(x) = n$, wenn $\phi(m)$ die Anzahl derjenigen Zahlen bezeichnet, welche relativ prim zu m und kleiner als m sind. (Progr.) Wien, 1901. 8vo. 17 pp.

REISENHOFER (R.). Die sphärischen Kegelschnitte. (Progr.) Kremsier, 1902. 8vo. 9 pp.

RIEMANN (B.). Gesammelte mathematische Werke. Nachträge, herausgegeben von M. Nöther und W. Wirtinger. Leipzig, Teubner, 1902. 8vo. 8 + 116 pp. M. 6.00

SAUERBECK (P.). Einleitung in die analytische Geometrie der höheren algebraischen Kurven nach den Methoden von J. P. de Gua de Malves. Ein Beitrag zur Kurvendiskussion. Leipzig, Teubner, 1902. 8vo. 6 + 166 pp. (Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, begründet von M. Cantor, Heft 15.) M. 8.00

—. Der Satz von de Gua über die Wendepunkte der Kurven dritter Ordnung. (Progr.) Reutlingen, 1901. 8vo. 8 pp.

SCHERING (E.). Gesammelte mathematische Werke. Herausgegeben von R. Haussner und K. Schering. (In 2 Bänden.) Vol. I. Berlin, Mayer & Müller, 1902. 4to. 8 + 412 pp. M. 25.00

SERRET (J. A.). Lehrbuch der Differential- und Integralrechnung. Mit Genehmigung des Verfassers deutsch bearbeitet von A. Harnack. 2te, durchgesehene Auflage, herausgegeben von G. Bohlmann. Vol. III. Lieferung 1: Differentialgleichungen, herausgegeben von G. Bohlmann und E. Zermelo. Leipzig, Teubner, 1903. 8vo. Pp. 1-304. M. 6.00

SNYDER (V.) and HUTCHINSON (J. I.). Differential and integral calculus. New York, American Book Co. [1902]. 12mo. 16 + 320 pp. Cloth. (Modern mathematical series.) \$2.50

STEININGER (T.). Studien zu Hesse's Analytischer Geometrie der geraden Linie, des Punktes und des Kreises in der Ebene. (Progr.) Rosenheim, 1902. 8vo. 38 pp., 1 plate.

STOLIAROV (N.). Collection of exercises in higher mathematics. Part I: Exercises in the differentiation of functions. Kiev, 1902. 8vo. 116 pp. (Russian.) M. 4.00

VALLÉE-POUSSIN (C. J. DE). Cours d'analyse infinitésimale. (En 2 volumes.) Vol. I. Paris, Gauthier-Villars, 1903. 8vo. 14 + 372 pp. Fr. 12.00

WALLENBERG (G.). See JAHRBUCH.

WERTHEIM (G.). Anfangsgründe der Zahlenlehre. Mit den Bildnissen von Fermat, Lagrange, Euler und Gauss. Braunschweig, Vieweg, 1902. 8vo. 12 + 427 pp. M. 9.00

WIRTINGER (W.). See RIEMANN (B.).

ZERMELO (E.). See SERRET (J. A.).

II. ELEMENTARY MATHEMATICS.

AGAPOV (D. V.). Arithmetical formulæ and their application to the solution of problems. 2 parts. Orenburg, 1902. 8vo. 64 + 79 pp. (Russian.) M. 4.00

ALASIA (C.). I complementi di geometria elementare. Milano, Hoepli, 1902. 16mo. 15 + 244 pp. (Manuali Hoepli.)

BAUSCHINGER (C. W.). See HAUCK (A. F.).

BERKENBUSCH (H.). Mathematisches Uebungsbuch für Realschulen, Realprogymnasien und Progymnasien. Aufgaben aus den Abgangsprüfungen sechsstufiger höherer Lehranstalten, zusammengestellt und mit Resultaten versehen. Berlin, Simion, 1902. 8vo. 5 + 136 + 41 pp. M. 2.80

BORK (H.). Mathematische Hauptsätze. Ausgabe für Gymnasien. Nach dem Tode des Verfassers herausgegeben von M. Nath. Teil I: Pensum der Unterstufe (bis zur Untersekunda einschliesslich). 4te, durchgesehene und den preussischen Lehrplänen von 1901 angepasste Auflage. Leipzig, Dürr, 1903. 8vo. 200 pp. Cloth. M. 2.10

— . Ausgabe für Realgymnasien und Oberrealschulen. Nach dem Tode des Verfassers herausgegeben von M. Nath. Teil I: Pensum der Unterstufe (bis zur Untersekunda einschliesslich). Nach der 3ten, vom Verfasser besorgten Auflage durchgesehene und den preussischen Lehrplänen von 1901 angepasste Ausgabe. Leipzig, Dürr, 1903. 8vo. 242 pp. Cloth. M. 2.50

BURILLO DE SANTIAGO (M.). Programa de primer curso de matemáticas que comprende nociones de aritmética y geometría. Madrid, 1902. 8vo. 21 pp. (Istituto general y técnico de San Isidro de Madrid.)

CERUELO Y OBISPO (J.). See MORENO REY (S.).

COOPER (G. H.). Elementary arithmetic of the octimal notation. San Francisco, Cal., Whitaker & Ray Co., 1902. 12mo. 70 pp. (Western mathematical series, No. 2.) \$0.25

CRAIG (E. S.). See MARCOU (C. A.).

F. (F.). Arithmétique (cours moyen). Paris, Poussielgue [1902]. 16mo. 305 pp.

FARISANO. Elementi di geometria descrittiva secondo i programmi dei reali istituti tecnici. Roma, Forzani, 1902. 8vo. 132 pp.

FISCHER (F.). See HAUCK (A. F.).

FOURREY (E.). Récréations arithmétiques. Paris, Nony, 1902. 8vo. Fr. 3.50

FUSS (K.). Resultate und Andeutungen zur Auflösung der Aufgaben aus der Buchstabenrechnung und Algebra. Für Schulen und zum Selbstunterricht bearbeitet. 5te, vermehrte und verbesserte Auflage. Nürnberg, Korn, 1902. 8vo. 7 + 176 pp. M. 1.60

- Sammlung von Konstruktions- und Rechenaufgaben aus der Planimetrie und Stereometrie. Mit vielen, vollständig gelösten Beispielen. Für den Schul- und Selbstunterricht bearbeitet. 3te, vermehrte und verbesserte Auflage. Nürnberg, Korn, 1902. 8vo. 8 + 252 pp. M. 2.50
- GEORGES (N. DE). See REBIÈRE (A.).
- GIDEON (E.). The model algebra; arranged for elementary schools. Philadelphia, Eldredge, 1902. 12mo. 149 pp. Cloth. \$0.60
- GLAUBER (R.). Die trigonometrische Aufgabe in Untersekunda. (Progr.) Erfurt, 1902. 8vo. 20 pp., 13 plates.
- HAUCK (A. F.) und HAUCK (H.). Lehrbuch der Arithmetik. Mit zahlreichen Beispielen und Übungsaufgaben. (In 3 Teilen.) Teil I, Abteilung 2: Für Real-, Gewerb- und Handelsschulen. 8te, durchgesehene und verbesserte Auflage, herausgegeben von C. W. Bauschinger. 4 + 191 pp. Teil II, Abteilung 1: Für Latein-, Real- und Handelsschulen. 6te, umgearbeitete Auflage, herausgegeben von F. Fischer. 6 + 228 pp. Nürnberg, Korn, 1903. Cloth. M. 5.30
- HEREDIA (G. DE). Lecciones de trigonometría, ajustada estrictamente al programa de ingreso en la Escuela central de ingenieros industriales. Madrid, Velasco, 1902. 8vo. 95 pp. Fr. 5.00
- HOČEVAR (F.). Lehrbuch der Arithmetik und Algebra nebst einer Sammlung von Übungsaufgaben für Oberrealschulen. Leipzig, Freytag, 1902. 8vo. 2 + 274 pp. Cloth. M. 3.60
- HOLZMANN (A.) und MASSINGER (R.). Geometrische Anschauungslehre, im Anschluss an den Lehrplan der badischen Realschulanstalten. (In 3 Teilen.) Teil I. Erster Abschnitt der ebenen Gebilde. Pensum der Klasse V. 3te Auflage. Karlsruhe, Reiff, 1902. 8vo. 32 pp. M. 0.60
- Teil II: Zweiter Abschnitt der ebenen Gebilde. Pensum der Klasse IV. 3te Auflage. Karlsruhe, Reiff, 1902. 8vo. 30 pp. M. 0.60
- HOLZMÜLLER (G.). Elemente der Stereometrie. Teil 4: Fortsetzung der schwierigeren Untersuchungen; Berechnung und stereometrische Darstellung von statischen Trägheits- und Centrifugalmomenten homogener Raumgebilde; Simpsonsche Regel; verallgemeinerte Schichtenformel; gewisse Zuordnungen und konforme Abbildungen im Dienste solcher Bestimmungen; Nachtrag über das Katenoid, seine Krümmungsverhältnisse und sphärische Abbildung und über seinen Zusammenhang mit der Gauss'schen Pseudosphäre und der Minimalschraubenregelfläche. Leipzig, Göschen, 1902. 8vo. 11 + 311 pp. M. 9.00
- JIMENEZ RUEDA (C.). Lecciones de geometría métrica. (In 2 volumes.) Madrid, Suárez, 1902. 8vo. 215 + 248 + 18 pp., 34 plates. Fr. 16.00
- JOCelyn (L. P.). An algebra for high schools and academies. Philadelphia, Butler, Sheldon & Co. [1902]. 12mo. 445 pp. Cloth. \$1.25
- KRYZHANOVSKY (V. A.). Solutions of 256 geometric exercises by A. Davidov. 3d edition, revised. Kiev, 1901. 8vo. 137 pp. (Russian.) M. 3.00

- MACÉ DE LÉPINAY (A.). See VACQUANT (C.).
- MARCOU (C. A.) and CRAIG (E. S.). Oxford questions in algebra; papers set in pass moderations; with answers. 2d series, from Michaelmas term 1891 to Trinity term 1899. London, Simpkin, 1902. 12mo. Boards. 2s. 6d.
- MARKE (C. I.). Mathematical questions and solutions from the *Educational Times*. Vol. II. New series. London, Hodgson, 1902. 8vo. Cloth. 6s. 6d.
- MARTÍNEZ GARCÍA (M.). Programa de geometría. Madrid, 1902. 8vo. 20 pp. (Instituto del Cardenal Cisneros.) Fr. 1.00
- . Programa de nociones y ejercicios de aritmética y geometría. Primer curso. Madrid, 1902. 8vo. 23 pp. Fr. 0.75
- . Nociones y ejercicios de aritmética y geometría. Madrid, 1902. 8vo. 185 pp. Fr. 6.00
- MARTINI ZUCCAGNI (A.). Trattato di algebra con i complementi di aritmetica razionale ad uso dei licei, secondo gli ultimi programmi governativi. Livorno, Giusti, 1903. 16mo. 12 + 328 pp. Fr. 2.60
- MASSINGER (R.). See HOLZMANN (A.).
- MOČNIK (F. VON). Lehrbuch der Arithmetik und Algebra nebst einer Aufgabensammlung für die oberen Klassen der Realschulen, bearbeitet von A. Neumann. 26ste, veränderte Auflage. Leipzig, Freytag, 1902. 8vo. 324 pp. Cloth. M. 3.80
- MORENO REY (S.) y CERUELO Y OBISPO (J.). Elementos de matemáticas. Trigonometría. 6a edición. Madrid, 1902. 8vo. 63 pp. Fr. 1.50
- NATH (M.). See BORK (H.).
- NEUMANN (A.). See MOČNIK (F. VON).
- NIEWENGLOWSKI (B.). Cours d'algèbre, à l'usage des élèves de la classe de mathématiques spéciales et des candidats à l'Ecole normale supérieure et à l'Ecole polytechnique. 5e édition. Paris, Colin, 1902. 8vo. 6 + 390 pp.
- PIROZHEIKOV (M. V.). See SERRET (J. A.).
- REBIÈRE (A.). Trigonometry, with exercises. Translated by N. de Georges. 3d edition. St. Petersburg, 1902. 8vo. 207 pp. (Russian.) M. 2.50
- ROEDER (H.). Der Koordinatenbegriff und einige Grundeigenschaften der Kegelschnitte. Zunächst eine Ergänzung zur Neubearbeitung der Planimetrie von Kambly. Zum Gebrauche an Gymnasien bearbeitet. 2te, vermehrte Auflage. Breslau, Hirt, 1902. 8vo. 64 pp. Boards. M. 0.60
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A. M.			A. B.		
Sat.,	Oct. 25,	1902, 11:00	Sat.,	Feb. 28, 1903, 11:00	
M.-Tu.,	Dec. 29-30,	" "	" "	April 25,	" "

The Meeting of December 29-30 will be the Annual Meeting of the Society for the Election of Officers and Delivery of the Presidential Address.

The Chicago Section will meet at the University of Chicago, January 2-8, 1903.

The San Francisco Section will meet at the University of California, December 20, 1902.

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ON THE TRANSFORMATION OF THE BOUNDARY IN THE CASE OF CONFORMAL MAPPING.

BY PROFESSOR W. F. OSGOOD.

(Read before the American Mathematical Society, December 30, 1902.)

THE theorem that the interior of any simply connected region may be mapped in a one-to-one manner and conformally on the interior of a circle was stated by Riemann in his doctor's dissertation and was proved by Schwarz and Neumann for the case that the boundary consists of a curve subject to certain restrictions. In the first volume of the *Transactions** I extended this theorem to the most general simply connected plane region. The generalization is remarkable in that the boundary of such a region need not be a curve, but may be a set of points some of which can not be approached along any curve lying wholly in the region. An extension of the definition of a simply connected region is here obviously necessary. Such a definition may be laid down as follows: By a simply connected region of a plane is meant a continuum such that the whole interior of any closed Jordan curve drawn in the region lies in the region. By a "Jordan curve" is meant a continuous curve without multiple points.

The further question presents itself as to whether boundary points of the region are carried over continuously into points of the circumference of the circle. This question I am able to answer by means of the following theorems. The given simply connected region is denoted by S and it is mapped conformally on the interior of the circle $\mathfrak{S}$.

THEOREM I. *Let A be a point of the boundary of S which can be approached along a curve C lying wholly in S . Then the image of C in $\mathfrak{S}$ is a curve $\mathfrak{C}$ abutting on a single point $\mathfrak{A}$ of the circumference of $\mathfrak{S}$; so that, if the point P approaches A along C , the image $\mathfrak{P}$ of P in $\mathfrak{S}$ will approach $\mathfrak{A}$ along $\mathfrak{C}$.*

THEOREM II. *Let C' be a second curve of S also leading to A , but meeting C only in the initial point of C and in the point*

* "On the existence of the Green's function for the most general simply connected plane region," *Transactions Amer. Math. Soc.*, vol. I (1900), p. 310.

A. Then the image of C in $\mathfrak{S}$ will be a curve $\mathfrak{C}'$ abutting on a point $\mathfrak{A}'$ of the circumference of $\mathfrak{S}$. The necessary and sufficient condition that $\mathfrak{A}'$ coincide with $\mathfrak{A}$ is that the closed curve consisting of the two arcs C, C' may be drawn together continuously to the point A without passing out of the region S ; in other words, that this closed curve shall contain in its interior only points of S .

Note that S consists exclusively of interior points, the boundary not being counted as belonging to the region.

THEOREM III. *If A, A' are two distinct points of the boundary of S which can be approached along a curve C or C' respectively, then the corresponding points $\mathfrak{A}, \mathfrak{A}'$ on the circumference of $\mathfrak{S}$ will be distinct.*

THEOREM IV. *Let B be a point of the boundary of S , such that a set of points $Q_1, Q_2, \dots$ can be chosen in S which have B as their sole limiting point:*

$$\lim_{n \rightarrow \infty} Q_n = B;$$

but through which no curve can be passed, lying wholly in S and such that a point P describing this curve will approach B as its limit. Suppose furthermore that a curve D can be drawn in S passing through the points Q_n and having but one limiting point A on the boundary of S which can be approached along a curve C , cutting D in every neighborhood of A . Then if the point P describe D , the image $\mathfrak{P}$ of P in $\mathfrak{S}$ will approach the point $\mathfrak{A}$ of the boundary of $\mathfrak{S}$ which corresponds to A .

THEOREM V. *In particular, let S be a region having no boundary points of the kind B considered in Theorem IV. Then to each point of the circumference of $\mathfrak{S}$ will correspond a single point of the boundary of S in this sense that if a point $\mathfrak{P}$ approach a point $\mathfrak{A}$ of this circumference along a curve of $\mathfrak{S}$, the image P of $\mathfrak{P}$ in S will approach a point A of the boundary of S , and A will be the same point for all curves of $\mathfrak{S}$ leading to $\mathfrak{A}$. But to distinct points $\mathfrak{A}, \mathfrak{A}'$ may correspond the same point A .*

The boundary of S in this case consists of a curve whose coördinates may be represented as single valued continuous functions of a parameter t

$$x = f(t), \quad y = \phi(t),$$

where t may be chosen as the arc of the circumference of $\mathfrak{S}$, and $f(t), \phi(t)$ are periodic with the length of the circumference of $\mathfrak{S}$ as period.

If furthermore it is impossible to draw a Jordan curve all of whose points but one lie in S and which contains in its interior points of the boundary of S , then the boundary points of S will form a Jordan curve.

The result stated in the latter part of this theorem was recently published by Schoenflies.* In the general case the boundary may be considered as forming a generalization of the Jordan curve; for, although the boundary may pass through a given point more than once, it does not cut itself, and it forms the boundary of a simply connected region.

The method on which the proof of these theorems rests was suggested by physical evidence. The conformal map of S on $\mathbb{C}$ is effected by the Green's function g of the region S , with pole at O , the existence of g having been established in my former paper (*l. c.*). This function may be interpreted in the well-known manner as the temperature in a certain flow of heat or as the potential in the analogous flow of electricity, the source being at O . The function h conjugate to g can be employed to measure the flux across a given curve; in fact, if c denote an arc drawn in S , the flux across c is proportional to the difference in the values of h at the extremities of c . I was led to the theorems stated above through the following

PHYSICAL LAW.—*Let a fixed circle be drawn in S with O as centre. Let c be an open curve of length l drawn in S , but lying outside the fixed circle. Let the maximum numerical value of the flux across c (or the upper limit, in case no maximum exists), when c assumes all possible forms and positions, be denoted by $f(l)$. Then*

$$\lim_{l=0} f(l) = 0.$$

A rigorous proof of this law may be given by means of the theory of the logarithmic potential function, and proofs of all the theorems here stated will appear later.

HARVARD UNIVERSITY, CAMBRIDGE,
17 December, 1902.

* "Ueber einen grundlegenden Satz der Analysis situs," *Göttinger Nachrichten*, 1902, Heft 2.

ON THE QUINTIC SCROLL HAVING THREE DOUBLE CONICS.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society, October 25, 1902.)

IN his paper on quintic scrolls,* Professor Schwarz mentioned that such a scroll exists having three double conics (type IX of those of genus 0). It is now proposed to derive the equation of this surface and to study some of its properties.

A scroll having two double conics will in general be of order 8 and genus 1. In order to generate a quintic scroll the conics must therefore intersect in two points, one of which is a simple self-corresponding point of the (2, 2) correspondence between the points of the two conics, and the other must be a double self-corresponding point. Hence every such quintic scroll must be unicursal. Let the equations of the two conics be

$$\begin{aligned} c_1: \quad & z = 0, \quad x^2 - yw = 0; \\ c_2: \quad & x = 0, \quad z^2 - yw = 0, \end{aligned}$$

respectively. The points of each may be expressed parametrically thus:

$$\begin{aligned} c_1: \quad & x = \lambda, \quad y = \lambda^2, \quad z = 0, \quad w = 1; \\ c_2: \quad & x = 0, \quad y = \mu^2, \quad z = \mu, \quad w = 1. \end{aligned}$$

The equations of a generator which joins the variable point λ to the variable point μ may be written

$$(1) \quad \mu x + \lambda z - \lambda \mu w = 0, \quad \lambda x - y + \mu z = 0.$$

The line (1) will generate a unicursal quintic scroll having c_1, c_2 for double conics when λ, μ are connected by a relation of the form described. Since the complete double curve of a unicursal quintic is of order 6 it follows that the residual curve is of order 2. Double generators are impossible since the double element in the (2, 2) correspondence is self-corresponding.

* Ueber die geradlinigen Flächen fünften Grades, *Crelle's Journal*, vol. 67.

Two rectilinear directrices can not exist on a scroll having other nodal curves; hence the residual nodal curve is a conic c_3 . Since a similar correspondence exists between the new conic and each of the given ones, it follows that the three conics must pass through one common point. No scroll can exist having three double conics which intersect in three distinct pairs of points.

§ 1. *Three Double Conics Having One Common Point.*

Let $(0, 0, 0, 1)$ be the point common to the three conics.

Let c_1 intersect c_3 in $(\alpha, \alpha^2, 0, 1)$, and c_2 intersect c_3 in $(0, \gamma^2, \gamma, 1)$.

The conic c_3 lies in the plane

$$(2) \quad \alpha x - y + \gamma z = 0,$$

and may be expressed as the intersection of (2) and the most general quadric passing through the three given points. The values of $x : y : z : w$ as determined from (1) and (2) are now to be substituted in the equation of the quadric. The result is the desired (2, 2) correspondence between λ, μ . Finally, the result of eliminating λ, μ between this equation and those of a generator is the desired equation of the surface.

The correspondence is of the form

$$(3) \quad \lambda^2 - \mu^2 \kappa + m\lambda\mu + \gamma(\kappa + 1)\mu - \alpha(\kappa + 1)\lambda = 0.$$

It has a double element at (∞, ∞) and a simple self-corresponding element at $(0, 0)$, hence the point common to all three conics cannot be a double point. Similarly, the point α is a double point in the correspondence between c_1 and c_3 , and γ is a double point in the correspondence between c_2 and c_3 . Since the plane of each double conic must contain one generator it follows that three generators pass through the point $(0, 0, 0, 1)$. The equations of these generators are

$$x = 0, \quad \kappa y = (\gamma\kappa + \gamma)z;$$

$$y = 0, \quad \alpha x + \gamma z = 0;$$

$$z = 0, \quad y = (\alpha\kappa + \alpha)x.$$

When $\kappa = 1$ they all lie in the plane

$$y = (\alpha\kappa + \alpha)x + (\gamma\kappa + \gamma)z.$$

The residual curve in the plane of the three generators is a conic * c_4 ; it does not pass through the triple point. This is the only simple conic which can lie on the surface.

Moreover, the scroll is contained in a linear complex and has one algebraic asymptotic line of order 8. If in addition $m = 0$, the complex becomes special and the simple conic breaks up into a simple directrix and a generator. Every plane through the directrix will cut four generators from the surface. The directrix does not intersect any of the double conics, hence the three points in which the three generators from $(0, 0, 0, 1)$ meet the conics again are collinear. The scroll can now be generated by a particular $(1, 2)$ correspondence between the points of the conic and the straight line.

The equations of the directrix are

$$w = 0, \quad y = 2ax + 2yz,$$

and the fourth generator in the plane $y = 2ax + 2yz$ is defined by

$$\gamma x + \alpha z = 2\alpha y.$$

This generator connects the three points in which the generators through $(0, 0, 0, 1)$ cut the double conics again. The special asymptotic line of order 8 is now degraded into the directrix of the complex.

When a unicursal quintic scroll has a double conic the residual nodal curve is in general a twisted quartic with two apparent double points. In general a straight line and a quartic having two apparent double points will determine a scroll of order 8 by the condition that each generator is to cut the line once and the quartic twice. The straight line is a double line on the surface and the quartic curve is a triple line. When the surface is reducible and one factor is a quintic having the line as simple directrix and the quartic for double curve, the other component is a cubic having the straight line and the quartic as simple directrices.

Professor Schwarz remarks that the same thing happens if the quartic be replaced by two conics which have two points of intersection. But such is not the case. Let c_1, c_2 be the conics and l the line. Let c_1, c_2 intersect in $W \equiv (0, 0, 0, 1)$ and in $Y \equiv (0, 1, 0, 0)$. The scroll determined by the common sec-

* Mr. C. H. Sisam has recently proved that a simple conic lies on the general scroll. Its plane does not pass through the triple point.

cants of two conics and a line is of order 8, less 1 for each point in which the conics intersect, since the pencils Wl , Yl belong to the system. Our scroll is therefore of order 6 and genus 1, having c_1 , c_2 , l for double lines. The residual nodal curve is a twisted cubic which is met once by each generator.

If, however, the line l lies in one of the common tangent planes of the two conics the surface will be of order 5, as may be shown as follows: A generator (1) will cut the line

$$A_1x + B_1y + C_1z + D_1w = 0, \quad A_2x + B_2y + C_2z + D_2w = 0$$

if

$$\lambda^2\mu(AB) + \lambda\mu^2(BC) + \dots + \lambda(CD) + \mu(AD) = 0.$$

This is in general a (2, 2) correspondence of genus 1 with two simple self-corresponding elements at the points of intersection of the conics. If the scroll is to be a quintic, one of these points must be a double element. This condition requires either that the coefficients of $\lambda\mu^2$ and $\lambda^2\mu$ shall vanish, or that the coefficients of λ and μ shall vanish. In the first case the line l lies in the plane $w = 0$, and in the second case l lies in the plane $y = 0$. But these are the common tangent planes of c_1 and c_2 at W , Y . It can easily be verified independently that when $\kappa = 1$ and $m = 0$ the tangent planes of c_1 and c_2 in $(0, 1, 0, 0)$, the tangent plane of c_1 and c_3 in $(0, \alpha^2, \alpha, 1)$, and the tangent plane of c_2 and c_3 in $(\gamma, \gamma^2, 0, 1)$ all pass through the simple directrix.

The only case in which a straight line and two conics which intersect in two points can define a cubic scroll is when the (2, 2) correspondence is factorable, for such a surface is determined by a (1, 1) correspondence between a line and a conic. For example, the conics c_1 , c_2 and the line $y = 0$, $x + z = w$ define a reducible (2, 2) correspondence between λ , μ and all lie on the same cubic scroll.

The equation of our quintic, obtained by eliminating λ , μ between (1) and (3), may be written

$$\begin{vmatrix} x^2 - \kappa x^2 - m x z & y + n x & l z - \kappa y & 0 \\ 0 & y + n x & \kappa x y + m y z + n z^2 & l z - \kappa y \\ w & 1 & x & 0 \\ 0 & x & z^2 - y w & x \end{vmatrix} = 0,$$

in which $l \equiv \gamma(\kappa + 1)$, $n \equiv -a(\kappa + 1)$.

For the equation of the second surface, having a linear directrix, put $\kappa = 1$, $m = 0$.

§ 2. *Three Double Conics Having Two Common Points.*

Now consider the case in which the third double conic passes through both points of intersection of c_1 , c_2 . Let

$$x = pz$$

be the equation of its plane. The equation of the general conic lying in this plane and passing through $(0, 0, 0, 1)$, $(0, 1, 0, 0)$ may be written

$$Ax^2 + Bzy + Cyw + Dzw = 0,$$

from which the $(2, 2)$ correspondence becomes

$$B\lambda^2\mu(p+1) + C\lambda^2p + (A + p^2C + C)\lambda\mu + pC\mu^2 \\ + pD\mu + D\lambda = 0.$$

In order that the correspondence have a double element at one point of intersection of the two conics, and a simple self corresponding element at the other, either $B = 0$ or $D = 0$. This requires that the third conic must touch the common tangent plane of c_1 , c_2 at one of their points of intersection. Let $B = 0$. Then the new conic touches the plane $w = 0$ at $(0, 1, 0, 0)$.

The reduced form of the correspondence now becomes

$$\lambda^2 + \mu^2 + m'\lambda\mu + n'\lambda + l'\mu = 0$$

and the analytic procedure is identical with that of the preceding case.

The generators which lie in the planes of the three double conics are respectively

$$\begin{aligned} x &= 0, & y + l'z &= 0; \\ y &= 0, & x - pz &= 0; \\ z &= 0, & y + n'x &= 0; \end{aligned}$$

since $\kappa = 1$ they cannot lie in one plane. The scroll also contains a simple conic which cuts each of the double conics once. The lines which cut three conics related like c_1 , c_2 and the simple conic will in general generate a scroll of order

8. If one scroll be of order 5, the other will be a cubic and not another quintic, as stated by Schwarz.

In this case the surface does not belong to a linear complex; in particular, therefore, *when a quintic scroll has three coaxial double conics it cannot have a rectilinear directrix.*

§ 3. Two Double Conics Touching Each Other.

Any two conics which touch each other but which lie in different planes may be expressed by the equations

$$\begin{aligned} c'_1: \quad & z = 0, \quad y^2 - xv = 0; \\ c'_2: \quad & x = 0, \quad y^2 - wz = 0. \end{aligned}$$

In parametric representation these become

$$\begin{aligned} c'_1: \quad & x = 1, \quad y = \mu, \quad z = 0, \quad w = \mu^2; \\ c'_2: \quad & x = 0, \quad y = \lambda, \quad z = 1, \quad w = \lambda^2. \end{aligned}$$

The equations of the line joining λ to μ are

$$\mu x + \lambda z - y = 0, \quad \mu^2 x + \lambda^2 z - w = 0.$$

The most general (2, 2) relation between λ, μ which will define a quintic scroll is given by

$$g\lambda^2 + f\mu^2 - \lambda\mu(f + g) + h = 0.$$

The surface has a third double conic c'_3 defined by

$$y = 0, \quad h x z + g x w - f z w = 0.$$

The equation of the surface may be written

$$\begin{vmatrix} g & -(f+g)y & hx - fw & 0 \\ 0 & g(x+z) & -(f+g)y & hx - fw \\ z & -2yz & y^2 - wx & 0 \\ 0 & z(x+z) & -2yz & y^2 - wx \end{vmatrix} = 0.$$

The common tangent to c'_1, c'_2 is a generator. The generator in the plane $y = 0$ is $x + z = 0$. The curve of section made by the plane $x + z = 0$ is

$$zy^2[(f+g)(y^2 + wz) + 2z(hz - fw)] = 0;$$

hence the line $y = 0, x + z = 0$ is a torsal generator and $x + z = 0$ is the torsal plane. The simple conic passes through the point of intersection of the three double conics and touches the generator $x = 0, y = 0$.

If $f + g = 0$ the conic will be factorable. The rectilinear directrix is

$$x + z = 0, \quad hz - fw = 0.$$

If the simple conic be required to intersect c'_3 in a second point, the former must break up.

When the scroll belongs to a linear complex, the complex must be special. A quintic scroll which has three double conics which touch the same line at the same point, does not exist.

Every quintic scroll having three double conics contains either a simple conic or a simple directrix. If in the former case the planes of all four conics pass through the same point, then the plane of the simple conic contains three generators passing through that point.

CORNELL UNIVERSITY,
October 10, 1902.

SURFACES REFERRED TO THEIR LINES OF LENGTH ZERO.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, October 25, 1902.)

CONSIDER a surface S referred to a general system of parametric lines, $u = \text{const.}, v = \text{const.}$ Denote by x, y, z , the cartesian coördinates of a point on S and X, Y, Z , the direction cosines of the normal to the surface. As usual, we put

$$(1) \quad E = \sum \left(\frac{\partial x}{\partial u} \right)^2, \quad F = \sum \frac{\partial x}{\partial u} \frac{\partial x}{\partial v}, \quad G = \sum \left(\frac{\partial x}{\partial v} \right)^2,$$

and

$$(2) \quad D = \sum X \frac{\partial^2 x}{\partial u^2}, \quad D' = \sum X \frac{\partial^2 x}{\partial u \partial v}, \quad D'' = \sum X \frac{\partial^2 x}{\partial v^2}.$$

When the parameter curves are the lines of length zero,

$$E = G = 0,$$

and the Codazzi equations * reduce to the simple forms

$$(3) \quad \begin{aligned} \frac{\partial D}{\partial v} - D' \frac{\partial}{\partial u} \log \frac{D'}{F} &= 0, \\ \frac{\partial D'}{\partial u} - D' \frac{\partial}{\partial v} \log \frac{D'}{F} &= 0. \end{aligned}$$

The expression for the mean curvature † becomes

$$(4) \quad \frac{1}{2} \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) = - \frac{D'}{F}.$$

From (3) it follows that when S is a surface of constant mean curvature, D is a function of u alone and D' of v alone. Conversely, when this condition is satisfied, either $D' = 0$ or D'/F is constant. In either case the mean curvature is constant; in the former, S is a minimal surface. Hence

THEOREM I. *Surfaces of constant mean curvature are characterized by the property that, when the parametric lines are of length zero, D is a function of u alone and D' of v alone.*

When any surface is referred to its lines of length zero, the directions of the lines of curvature are given by the equation ‡

$$(5) \quad Ddu^2 - D'dv^2 = 0.$$

In consequence of the above results we have

THEOREM II. *The lines of curvature on a surface of constant mean curvature are given by quadratures when the surface is referred to its lines of curvature.*

By Bonnet's theorem § we know that there is associated with every surface of constant mean curvature, different from zero, a parallel surface of constant total curvature and conversely. Moreover, it can be seen from the general relations || which

* Bianchi, Lezioni, p. 91.

† *Ibid.*, p. 104.

‡ *Ibid.*, p. 99.

§ Darboux, Leçons, vol. 3., p. 375.

|| Knoblauch, Einleitung in die allgemeine Theorie der krummen Flächen, p. 234-6.

obtain between the fundamental coefficients of a surface and its parallel that lines of length zero and lines of curvature upon a surface of constant mean curvature correspond to asymptotic lines and lines of curvature respectively upon the parallel surface of constant total curvature. Hence

THEOREM III. *The lines of curvature on a surface of constant total curvature are given by quadratures, when the surface is referred to its asymptotic lines.*

Theorem II is not restricted to surfaces of constant mean curvature, but, as we shall see, is true of all isothermic surfaces. Let S be an isothermic surface; then its linear element can be brought to the form

$$(6) \quad ds^2 = \lambda(du_1^2 + dv_1^2),$$

where λ is a function of u_1 and v_1 , and the curves $u_1 = \text{const.}$, $v_1 = \text{const.}$ are the lines of curvature. Now the lines of length zero are given by

$$du_1^2 + dv_1^2 = 0,$$

that is,

$$(7) \quad u_1 + iv_1 = 2u, \quad u_1 - iv_1 = 2v,$$

where u and v are the parameters of the lines of length zero. From (7) it follows that, if the surface is referred to lines of length zero, the equation of the lines of curvature is

$$du^2 - dv^2 = 0,$$

and for a general choice of parameters this would be

$$Udu^2 - Vdv^2 = 0,$$

where U is a function of u alone and V is a function of v alone. Comparing this equation with equation (5) we get

$$(8) \quad \frac{D}{D'} = \frac{U}{V}.$$

Hence

THEOREM IV. *When an isothermic surface is referred to its lines of length zero, the lines of curvature are given by quadratures.*

And conversely

THEOREM V. *When a surface is referred to its lines of length zero and the condition (8) is satisfied, the surface is isothermic.*

In consequence of the preceding results the equation whose

solution gives all surfaces of constant mean curvature can be reduced to a simple form. By a suitable choice of parameters we have $D = D' = 1$, and there is no loss in assuming that the square of the mean curvature is unity. When these values are substituted in the Gauss equation,* it reduces to

$$\frac{1}{F} \frac{\partial^2 \log F}{\partial u \partial v} = \frac{1}{F^2} - 1.$$

If we put $\log F = \theta$, this equation becomes

$$\frac{\partial^2 \theta}{\partial u \partial v} = e^{-\theta} - e^{\theta}.$$

When a solution of this equation is known, all of the fundamental coefficients of the surface are given.

PRINCETON, N. J.,
September 20, 1902.

SUPPLEMENTARY NOTE ON THE CALCULUS OF VARIATIONS.

BY DR. E. E. HEDRICK.

IN a recent paper in the BULLETIN,† the present writer stated the theorem: "The condition $\partial^2 f / \partial y'^2 > 0$ for all x, y on the curve C , and for all y' considered, is a sufficient condition for a minimum" (of the integral $\int f(x, y, y') dx$ along the curve C), provided that certain preliminary requirements are satisfied. This theorem stands in apparent contradiction with an example given by Professor Bolza;‡ and while the contradiction is *only* apparent, it seems fitting to point out clearly the actual agreement of my results with those of Professor Bolza.

In the article mentioned I have required (page 11) that the integral $\int f(x, y, y') dx$, shall be considered only in a region R (of the x, y, y' space), such that the integrand $f(x, y, y')$ is

* Bianchi, *l. c.*, p. 67.

† "On the sufficient conditions in the calculus of variations," BULLETIN, vol. 9 (2), no. 1 (Oct., 1902), p. 15.

‡ "Some instructive examples in the calculus of variations," BULLETIN, vol. 9 (2), no. 1 (Oct., 1902), Example II, p. 9.

analytic at each point of R , i. e., for all sets of values of x , y , y' in R . A region is here a closed manifold of points, i. e., it possesses a boundary, so that a point upon the boundary of R belongs to the region. Hence for the reasoning throughout it is necessary to choose such a region R . It is true that this restriction is by no means necessary for most of the results of the paper, but without it some of the proofs, and in particular the above theorem, would have to be modified. It is to be noticed, however, that we may choose an *unbounded* region R_1 (i. e., exclusive of the boundary) provided R_1 lies wholly within a region R such as is required above, and that all the results of the paper hold when the given integral is considered only for points *inside of* R_1 , or in particular *inside of* R itself. In other words, the results given hold in any region (inclusive or exclusive of the boundary) provided the points for which the integrand $f(x, y, y')$ is not analytic be excluded from the region (i. e., from its interior and from its boundary).

In Professor Bolza's example, I would then, for the purposes of my paper, *first* choose as my region R the region

$$R: \quad |x| \leq A; \quad |y| \leq A; \quad |y'| \leq A.$$

Any positive number A having been chosen for the example once for all, the investigation of minima depends on our ability to find, *secondly*, a value for δ so small that the supposed solution renders the integral value less than along any other curve for which

$$|\eta(x)| < \delta, \quad x_0 \leq x \leq x_1.$$

It is seen at once that any *order of choice* of these quantities A and δ corresponds exactly to the same order of choice of the h and k of Professor Bolza's article, respectively; whereas the order of choice necessary for the considerations of my paper is exactly opposite to that made by Professor Bolza. For, from my standpoint, k cannot be *first* chosen and then kept fixed while h (*secondly*) approaches zero, as is necessary for Professor Bolza's argument. A choice of h *after* k has been chosen and fixed, necessitates a choice of $A \cong k/h$. This I would regard as the first step, after which I would have δ at my disposal, which would necessitate a re-choice of k , if δ be chosen less than k . It readily follows that, from my standpoint, the conclusion $\Delta I < 0$ cannot be drawn, and that the X axis actually renders

the integral in question a strong minimum,* in the above region R , i. e., in any region of values of x, y, y' from the interior and boundary of which the values $x = \infty, y = \infty, y' = \infty$ are excluded. Accordingly the example cited, as also Professor Bolza's theorem at the top of page 9, are perfectly reconcilable with my theorem quoted above.

The integrand in question is discontinuous for infinite values of either x or y or y' , and in fact becomes infinite to the second order for $y' = \infty$. It may be pointed out that the same apparent contradiction will arise in the case of any integral whose integrand becomes infinite to the second order for any value of y' (not necessarily $y' = \infty$); but again the contradiction is only apparent. It might be questioned whether simple ordinary integrals, such as the integral of length, $\int \sqrt{1 + y'^2} dx$, do not present similar phenomena, so that the results of my paper would be restricted in their application, in all ordinary examples, to a closed region (of the x, y, y' space) in which the integrand is analytic. Such are indeed the formal requirements of the paper. But it may be noted that if the transformed integral

$$\int f\left(x, y, \frac{1}{x'}\right) x' dy,$$

where $x' \equiv dx/dy$, be analytic for the directions previously excluded ($y' = a, x' = 1/a$), then the whole reasoning applies to the comparison curves previously excluded, by a simple process of transformation and subdivision of the curve, at least provided that the comparison curves in question satisfy the simple requirement of not being parallel to each axis an infinite number of times in any interval.

And further, the requirement made on page 11 is by no means necessary for the proof of most of the theorems. In view, however, of the aims of the article, it was deemed advisable to include this restriction, in order to simplify the proofs, and in order to arrive at the theorem mentioned above without too extended a discussion of possible alternate restrictions.

E. R. HEDRICK.

SHEFFIELD SCIENTIFIC SCHOOL,
November, 1902.

* The distinction between strong and weak minima holds as usual because the slope restriction $|y'| \leq A$ was made before any particular extremal was found, and does not regard any particular extremal.

THE SYNTHETIC TREATMENT OF CONICS AT THE PRESENT TIME.

BY DR. E. B. WILSON.

(Read before the American Mathematical Society, December 29, 1902.)

IN the August-September number of the *Jahresbericht der Deutschen Mathematiker-Vereinigung* for the year 1902 Professor Reye has printed an address, delivered on entering upon the rectorship of the university at Strassburg, on "Synthetic geometry in ancient and modern times," in which he sketches in a general manner, as the nature of his audience necessitated, the differences between the methods and points of view in vogue in geometry some two thousand years ago and at the present day. The range of the present essay is by no means so extended. It covers merely the different methods which are available at the present time for the development of the elements of synthetic geometry—that is the theory of the conic sections and its relation to poles and polars.

There is one method of treatment which was much in use, the world over, half a century ago. At the present time this method is perhaps best represented by Cremona's *Projective Geometry*, translated into English by Leudesdorf and published by the Oxford University Press, or by Russell's *Pure Geometry* from the same press. This treatment depends essentially upon the conception and numerical measure of the double or cross ratio. It starts from the knowledge the student already possesses of Euclid's *Elements*. This was the method of Cremona, of Steiner, and of Chasles, as set forth in their books. That was fifty years ago.

In the year 1866 Professor Reye, mixing with a bold and somewhat original hand the newest ideas of von Staudt with those of Steiner and Poncelet, older and better known, published his *Geometrie der Lage* which owing to its incomparably clear and pedagogic style has been the mainstay of synthetic projective geometry these many years, slowly displacing the method of Cremona, and somewhat obscuring the elegance of von Staudt's.

At present, to judge from the text-books upon the market and the announcements of courses in the universities, the Ital-

ians have not only forsaken Cremona but are persistently edging by Reye toward the long neglected von Staudt; the Germans, pulled with almost equal forces backward by Steiner and forward by von Staudt, still cling to the intermediate, Reye. In France, where no one studies that which there is not some one great genius to teach, synthetic geometry in its higher aspects has seemingly perished by the way with Chasles—for what was it to Hermite and Bertrand or is it now to Darboux and Picard? In England and America Cremona is still much followed, and when not Cremona, then Reye.

In view of these facts, no more will be said of the metric, cross ratio method of developing synthetic geometry. That method caught its fatal chill when von Staudt published his book in 1847. It has been shaking ever since. Some day it will die. Not long hence even the fondest of us must recognize it as quite a thing of the past.

The newer methods are all founded ultimately upon the work of von Staudt, the essential part of which appeared in 1847 as has been mentioned, although the addition of three supplements stretched across the following thirteen years. These methods are more powerful and beautiful because they deal more directly with the ideas necessarily involved in synthetic projective geometry. They are more natural and satisfactory from a purely geometric standpoint. They are more compact and at the same time they are more far-reaching. In point of fact they are all nothing but variations on von Staudt's original, according as it is more or less fully adhered to, according as a less or a greater amount of the earlier methods are interwoven with it.

The chief variations, those which we shall discuss here, are three. We shall designate them as follows:

1° The extreme, which may be seen expounded in Reye's "Geometrie der Lage," 4th edition, volume I, Leipzig, or in Holgate's translation, New York.

2° The intermediate of which an excellent and elementary exposition is given in Böger's "Ebene Geometrie der Lage," Leipzig, or combined with the geometry of space in Sannia's "Geometria proiettiva," Naples.

3° The slight variation, or none at all, such as is presented in Enriques's most excellent "Geometria proiettiva," Bologna, or to a much less extent in a paper in the *Annals of Mathematics*, second series, volume 2, by Miss C. A. Scott.

By a glance at the books mentioned, these three methods may be seen to have in common a certain considerable number of elementary ideas and theorems; namely, the law of duality, the theorem of Desargues, the construction of harmonic elements, the theory of the projective relations between the points in two ranges or in the same range counted twice. These are explained perhaps most simply and succinctly by Böger. To them may be added, if desired, the elements of the theory of collineations and correlations between the points and lines of two planes or of the same plane counted twice.

From this point on, the three methods diverge. In the first a conic is defined as the locus of the intersection of corresponding rays in two projective, but not perspective, pencils. The usual theorems concerning inscribed and circumscribed quadrilaterals, Pascal's and Brianchon's theorems, and so forth, are then proved. The polar of a point with respect to a conic is defined by means of the well known harmonic property. The theory of poles and polars follows, completing the usual elements of conic sections from the synthetic standpoint.

The advantages of this method, the reasons why it holds its popularity so easily, are not hard to see. The processes seem direct. The construction of a conic is a pretty problem, a pretty exercise in draughting, for the student. (It is however by no means so pretty a problem to prove that a conic is independent of the two particular points upon it which may have been chosen as the vertices of the defining projective pencils.) A ruler and pencil are real, tangible things which are good for the student to use constantly. Moreover the idea of polar, introduced by means of the harmonic property, is easily grasped. The thing itself is easily constructed.

But there are disadvantages, too, of which one is that if the pole be outside the conic the construction for the polar yields only a segment of a line, only the part of the polar inside the conic. This is inconvenient. It introduces a sort of lack of symmetry which sticks out awkwardly in some proofs unless, drawing on the analogy furnished from analytic geometry, we make a bold application of the so-called principle of continuity. Again there is the awkward lack of symmetry mentioned in the parenthesis above. There are other disadvantages which will appear when the advantages of the third method are pointed out.

In the second method the conics are defined as before. To define the relations of pole and polar, the conception of an

involution upon a conic is used. Points $A, B, \dots; A', B', \dots$ upon a conic are said to be in involution if the relation between them is projective in such a manner that if A corresponds to A' , then A' corresponds mutually to A . It is shown that the lines which join corresponding points of an involution meet in a point P and that tangents drawn at corresponding points meet in a line p . The point P and line p are defined as pole and polar.

This method is not much better than the first. It is but a timorous intermediate between it and the third. It has nothing to recommend it but the drilling of the student in the ideas of projectivities and in the special case of involutions. These statements must not be construed as in any way detrimental to Böger's book, which is, on the contrary, far from tiresome and contains in its later portions a complete presentation of the newest ideas in adjoined and resultant involutions and in polar fields. In fact the standpoint, except at the commencement, is that of the third method of treatment.

The third method sets about matters in a wholly different way. Proceeding from the common ground before mentioned it lays great stress on the idea of projective relations between ranges or pencils and of collinear and correlative relations between two planes whether different or coincident. Here as in the first method the student may have plenty of use for his ruler and pencil. The subject only is different. Here he takes four points A, B, C, D and four other points A', B', C', D' and constructs corresponding points P and P' in the collineations thus defined, or he takes four lines a, b, c, d and constructs correlative points and lines P and p or inversely q and Q . These exercises are but preliminary. The important case is that of a *correlation which is involutory*, in which if P is correlative to p , then inversely p is correlative to P . Such an involutory correlation is called a polarity. The correlative points and lines are the poles and polars of the polarity.

As yet conics have not been defined. The definition is given as follows: The locus of those points in the polarity which lie upon their own polars (*if such points exist*) is a conic. Or the envelope of those lines in the polarity which pass through their own poles (*if such line exist*) is a conic. These two definitions lead to the same curve. The demonstrations that this curve is a conic according to the other definition of methods 1° and 2°, that the poles and polars with regard to the polarity are poles

and polars with regard to the conic, and all the usual theorems concerning conics and poles and polars follow immediately with great directness and simplicity.

The disadvantage which has been attributed especially to this method is its difficulty. There certainly is difficulty in the method either for the student or for the instructor. If the latter is content to follow hard in the footsteps of von Staudt or even directly after Enriques or C. A. Scott the student will have difficulty. This method has not been so trampled smooth, so laid out with problems as the older. But if the instructor will only apply himself to obtaining graded exercises for the student, if he will be content to go slowly and carefully, he will find that in the end he will have covered as much ground as easily and comprehensibly to the student, as if he had kept to either of the other methods.

And the advantages, whether on the score of beauty and power or on that of the insight afforded to the student into the subject of projective geometry, are alike great. For it must have been observed that the fundamental difference in the three methods is the relative importance in them of the idea of the transformation—the projective transformation, whether collineation or correlation—and in particular the involutory projective transformation. In the first method only so much knowledge of transformations is needed as will enable one to associate corresponding rays in two pencils, in the second the elementary properties of involutions on a conic are required, in the third the important fundamental notions of collineation and correlation are indispensable.

For the last thirty years, or to be more precise, since the publication of Klein's Erlangen Programm in 1872, there has been no reason why geometers should not recognize distinctly — although they have often been slow to do so — that the fundamental thing, next to the axiom, is the transformation or groups of transformations. In projective plane geometry, whether the treatment be analytic or synthetic, the important thing is the group of collineations G_3 plus the correlations Γ_3 . The conic, a projective concept in itself, belongs essentially to this group and may be treated most appropriately in connection with it. Von Staudt's great achievement was to see this and publish it in 1847, a quarter century before the appearance of Klein's Erlangen Programm. It is for the geometers who follow him so to arrange his method that it shall be easily acces-

sible to students without being changed so essentially as to lose its chief vantage points.

We may note in passing that in other fields than projective geometry as much if not greater sloth in adopting the transformation theory and its benefits is exhibited. How many a weary page of Euclid's elements could be brightened by the introduction of the theory of displacements in connection with polygons and polyhedra and of the theory of inversion in connection with circles and spheres. How many a useless proposition could be replaced by one full of promise for future applications.

To return to projective geometry. One advantage of von Staudt's method has been seen to be that it places in evidence the importance of the projective group. There is another advantage which might at first seem a disadvantage. It is this. The conic has been defined subject to its existence. There is a large class of polarities which yield no conic. Analytically they would yield an imaginary ellipse of the type

$$a^2x^2 + b^2y^2 + c^2 = 0.$$

But in these polarities a great number of theorems, real in every detail, hold just as in polarities which yield a real conic. For example, the theorem that the six vertices of two triangles self-conjugate in the same polarity lie on a conic is evidently real and valid for any sort of polarity. So with other theorems.

It is evident, therefore, that the third method, which in all essentials is von Staudt's, affords a treatment of polarities and consequently of conic sections which is independent of the reality or non-reality of those conic sections, provided only that the polar field defined by them is real. This means that we have a synthetic method for treating any conic section which would be representable analytically by an equation with real coefficients. Not often is a greater generality assumed for analytic methods.

The salient advantages of von Staudt's method are then these: First, insistence on treating projective geometry from the standpoint of the projective group. Second, ability to treat in a similar manner all conics, real or imaginary, which define real polar fields. The disadvantages are a supposed difficulty and indirectness. The latter does not exist and the former would reduce almost to nothing if half the attention were expended on the best method which has been spent on the worse.

I say the best method, referring to von Staudt's. It certainly seems at present the best not only of those known, but of any which may be invented. For what more or better could be expected of any purely synthetic method than that which is afforded by the two above mentioned salient advantages of this. It seems scarcely possible that any synthetic method should treat a more general class of conics or treat them in a more concise and germane manner.

ÉCOLE NORMALE SUPÉRIEURE, PARIS,
November, 1902.

BROWN'S LUNAR THEORY.

An Introductory Treatise on the Lunar Theory. By ERNEST W. BROWN, M.A. Cambridge, Eng., The University Press, 1896. xvi + 292 pp.

THE lunar theory, besides containing inherent difficulties of a very serious character, involves such a mass of details that it is only with a great deal of labor that one can get a satisfactory idea of it from the original memoirs. The essential relations and differences of the various methods are obscured as they come from the hands of their authors by the differences in the choice of variables and notations. In view of the intrinsic value of the subject and of its great importance as the best test of the newtonian law, and the fact that advances in it can hardly be hoped for from one who is not familiar with what has been done in the past, the desirability of a treatise starting at the very foundations, pointing out the difficulties which are encountered and the methods which have been used to overcome them, and giving the essentials of the most important processes employed by the various investigators without including the almost endless details, can easily be appreciated. This book has evidently been written to fill this need, and it may be said at once that Professor Brown has attained a very high degree of success.

The only other book of at all the same character is the third volume of Tisserand's *Mécanique céleste*, which is devoted to the theory of the motion of the moon. Tisserand's ideal is somewhat different, in that he aims to give a fairly complete account of all the lunar theories of particular merit which have

been developed. Moreover, his third volume is not complete in itself, as the results of the first two volumes are used so far as they are needed.

An idea of the contents and arrangement of Professor Brown's treatise can be obtained from the titles of the chapters, which are in their order: Force functions, The equations of motion, Undisturbed elliptic motion, Form of solution, The first approximation, Variation of arbitrary constants, The disturbing function, De Pontécoulant's method, The constants and their interpretation, The theory of Delaunay, The method of Hansen, Method with rectangular coördinates, The principal methods, Planetary and other disturbing influences. In the table of contents a complete enumeration of the topics treated is given, and this is supplemented by a general index, an index of authors, a reference table of notation, a general scheme of notation, and a comparative table of notation. From this it is seen that the work has been arranged so as to make its reading as simple as possible for the beginner, and to make it a valuable reference work for those who may be familiar with the subject.

The care with which the preliminary ground has been gone over has put the actual introduction of the perturbations due to the sun far along in the book. Thus, the reader finds himself compelled to read six chapters before he gets to that which he is continually expecting. It might, perhaps, have been better to have put the long chapter on the variation of constants after the chapters on de Pontécoulant's method and the constants and their interpretation. This would have remedied to a considerable extent that which in any case is apt to strain the patience of the reader. Aside from the possible modification just mentioned, the general plan of the book could scarcely be improved.

There are three essentially different types of lunar theory — that of de Pontécoulant, that of Delaunay, and that first developed by Hill, to which may perhaps be added that of Hansen as containing many features of more or less importance different from the others. That of de Pontécoulant and most of his predecessors consists in developing certain coördinates in periodic series of assumed form with the time or true anomaly as argument and determining the coefficients step by step as powers of the small parameters involved; that of Delaunay consists in applying the method of the variation of parameters in the

canonical form over and over in such a way as to remove the most important parts of the perturbative function ; that of Hill consists in finding very accurate particular solutions of the differential equations after the parts depending on the parallax of the sun, the eccentricity of the earth's orbit, and the latitude of the moon have been neglected, and then finding the deviations from this orbit due to general initial conditions and the neglected part of the perturbative function. The author has wisely treated these three types and the method of Hansen in detail, and explained the others briefly by their relations to them.

A number of typographical errors have crept in, but this could scarcely be avoided in a book involving so many complicated formulas. Those which have been noticed will cause the reader no trouble. There are some other points to be mentioned, having their origin for the most part in the imperfect state of the lunar theory from a mathematical standpoint rather than in faults of presentation of well-established methods. Thus, in the exposition of de Pontécoulant's method, which is somewhat different from the original yet fairly representative of methods of this type, the treatment of the constants of integration leaves something to be desired. The differential equations for motion in the plane of the ecliptic are, if $r = 1/u$ is the radius of the moon's orbit and v her longitude,

$$\frac{\frac{1}{2}a^2\left(\frac{1}{u^2}\right)}{dt^2} - u + \frac{1}{a} = m^2 P_1(u, v, e', t),$$

$$\frac{dv}{dt} - hu^2 = m^2 P_2(u, v, e', t).$$

a and h are constants depending on the initial conditions, m is the ratio of the mean motion of the sun to that of the moon, and e' is the eccentricity of the earth's orbit. If the right members are neglected, ordinary elliptic motion is obtained. In this, u is a periodic series whose coefficients are polynomials in the eccentricity of the moon's orbit, their structure depending uniquely on the form of the differential equations.

If the coördinates have the form $u = u_0 + \delta u$, $v = v_0 + \delta v$, where δu and δv are the perturbations, it is found after substituting in the differential equations that the main part of δu is defined by the equation

$$\frac{1}{n^2} \frac{d^2 \delta u}{dt^2} + \delta u = m^2 Q(t).$$

The general solution of this linear equation consists of two parts, the particular integral and the complementary function. The period of the complementary function is the same as that of the first periodic term in the elliptic motion. For this reason (see pages 50 and 117) the complementary function is omitted, or rather is supposed to be included in the elliptic term. This is of course allowable since the coefficient of the first elliptic term is arbitrary, but it is not evident that the coefficients of the higher multiples of the elliptic argument are related to this *modified* coefficient of the first term precisely as they are in undisturbed motion. This seems to be assumed tacitly, and it is to be regretted that the reason for it is not given if it is true. The fact is that there are but six arbitrary constants in the solutions of a system of differential equations of the sixth order, and it should be clearly shown how all of the constants can be expressed in terms of the six which are taken as the arbitraries. Some of the statements in Chapter VII might lead a careless reader to infer that many constants may be taken arbitrarily. It would have been clearer if the statement had been made that the constants $E + B + \dots$ and similar combinations should in the final results be set equal to a constant e_1 rather than that any of them should arbitrarily be set equal to zero. Such a method would express all of the constants in terms of six arbitraries; at the end transformations would be made so as to give six coefficients any desired form. This is the method clearly carried out by Delaunay.

Chapter VIII, which is devoted to a discussion of the meaning of the constants (that is, the coefficients of the principal terms) and the manner in which they are determined, is much to be commended. The treatment of the methods of Delaunay and Hansen is also quite satisfactory.

Chapter XI treats of the methods inaugurated by Hill in his celebrated *Researches*, and of the extensions to the terms of higher order which the author has himself carried out in numerous important memoirs. It is unfortunately called the "Method with rectangular coördinates." In the first place all of the earlier methods are named after their respective authors and there is no apparent reason why it should not be done here, especially in view of the fact that its whole spirit is

radically different from that of any previous method. In the second place the title adopted in no fundamental way characterizes the method. It is true that the differential equations are at first written in rectangular coördinates where the axes rotate uniformly ; but before the integration is begun a transformation is made to complex variables. It is clear that the rectangular coördinates are only incidental and that the final equations in the complex variables might be derived without using them at all. Hill's first problem was to find very exact particular solutions of the chief parts of the differential equations, and in doing this he used two important artifices. The first was to remove the explicit presence of the time in the differential equations by rotating the axes ; the second, to reduce the differential equations to the bilinear form by the use of complex variables in order that the task of finding the coefficients of the solutions might not be hopelessly involved. There is no trouble in finding the coefficients when the solutions of differential equations are developed as power series in the independent variable, or in parameters, but the general problem of finding the coefficients when the solutions are Fourier series presents great difficulties. Hill overcame them with rare ingenuity in this problem. If it is desired to give the method a title which characterizes it the method of the variational orbit might be used, but the reviewer suggests as being more appropriate that it be called the method of Hill, or the Hill-Brown method.

In developing the particular solutions of the restricted differential equations Brown, and Tisserand also, infer their form from the results of the earlier methods in the lunar theory. These methods, besides not having been proved to give convergent series, lead to expressions of the same form for general initial conditions. Now the particular solutions found depend upon particular initial conditions for their existence. Hill correctly inferred their character, if they exist, from the form of the differential equations ; and he proved their existence by the actual construction of certain nearly periodic orbits by mechanical quadratures and an appeal to the analytic continuity of curves when considered as functions of the initial values of the variables. It seems clear that Brown and Tisserand have taken a backward step in this matter, for it is evident that assumptions should be introduced as sparingly as possible, and that an ideal method is complete in itself.

The discussion of the terms which depend upon the ratio of the periods of the moon and sun and the first power of the eccentricity of the moon's orbit is in excellent shape for one to get from it a practical command of the ideas which Hill developed in his memoir in the *Acta Mathematica*. Yet from a theoretical point of view it would have been better to have proved the existence of the periodic solutions and their form, or at least to have referred to places where such proofs are given, rather than to have inferred their existence and character from earlier theories. The equations which give these terms are linear and non-homogeneous with periodic coefficients and the properties of their solutions are now well known. It might also have been remarked that there are other methods of finding the coefficients and the constant which determines the part of the motion of the moon's perigee which is of the first order in the eccentricity.

A very interesting phenomenon occurs in the literal expansions of the coefficients of the variational terms. If the expansions are made in powers of m (m equals about $\frac{1}{18}$), where m is the ratio of the mean motion of the sun to that of the moon, the series converge very slowly. But if the expansions are made in powers of m' where $m' = m/(1 - m/3)$ (therefore $m' = \frac{3}{8}$ nearly) the series converge with great rapidity. It is at first astonishing that so slight a change in the parameter should have such important results.

The most important parts of these coefficients come from the expansions of $(6 - 4m + m^2)^{-1}$. Hill inquires what value of α must be substituted in the equation $m' = m/(1 + \alpha m)$ so that the series in m' shall converge most rapidly. Hill says, and Brown repeats the statement, that it is easily found that α must be given the value $-\frac{1}{3}$. It is not stated just what is meant by rapidity of convergence, but if the remainder after a given number of terms is to be a minimum the statement is not perfectly correct, although $-\frac{1}{3}$ is undoubtedly the most convenient value to give α in the case in hand. If m had a value large enough the series would actually converge more slowly after the transformation.

The radius of convergence of the series in m is $\sqrt{6}$, and in m' , when $\alpha = -\frac{1}{3}$, $\sqrt{18}$. If m has such a value that $m' > \sqrt{3} m$ the new series will converge less rapidly than the old. The value of α which must be used to secure a minimum for the remainder depends upon both the numerical value of m and

the number of terms taken. Its value can be determined nearly from the formulas for the remainder limited by inequalities.

Let $f(m) = (6 - 4m + m^2)^{-1}$ and $\phi(m')$ be the same function after the substitution $m' = m/(1 + \alpha m)$. Let ρ represent the true radius of convergence of $f(m)$ when expanded as a power series in m , and ρ' the corresponding radius when $\phi(m')$ is expanded as a power series in m' . Let ρ_1 be a real positive number such that $\rho_1 < \rho$. Let M be the maximum value of the function $f(m)$ on the circumference of the circle whose radius is ρ_1 . As m takes all the values on the circumference of this circle m' will take all the values on the circumference of a circle whose radius may be represented by ρ'_1 . The maximum value of $\phi(m')$ on this circumference is also M . Let $R_n(m')$ be the remainder of the series in m' after the first n terms have been taken. Then $R_n(m')$ is limited by the inequality

$$R_n(m') < \frac{M \left(\frac{m'}{\rho'} \right)^n}{1 - \frac{m'}{\rho'}} \text{ for all } |m'| < \rho'_1.$$

Consider now the problem of making $R_n(m')$ a minimum. Let ρ_1 be taken as the numerical value of m ; hence M is defined by the given value of m and the form of the function $f(m)$, and is a constant with respect to the proposed transformation. Without an absolute minimum being insured because of the presence of the inequality, $R_n(m')$ will generally be smaller the smaller the fraction $(m'/\rho')^n/(1 - m'/\rho')$ is made. This fraction depends upon α as the arbitrary. Let it be represented by $F(\alpha)$. The necessary condition for a minimum is $\partial F(\alpha)/\partial \alpha = 0$. In order to compute this derivative, m' and ρ' must be expressed in terms of α . From the equation of transformation $m' = m/(1 + \alpha m)$ and therefore $m = m'/(1 - \alpha m')$ and

$$\begin{aligned} & (6 - 4m + m^2)^{-1} \\ &= (1 - \alpha m')^2 [6 - 4(1 + 3\alpha)m' + (1 + 4\alpha + 6\alpha^2)m'^2]^{-1}. \end{aligned}$$

The only singularities are poles and it follows at once that

$$\rho' = \frac{\sqrt{6}}{\sqrt{1 + 4\alpha + 6\alpha^2}}.$$

Now forming the partial derivative, casting out the common factors, and clearing of fractions it is found that

$$\begin{aligned} & 12n(1+3\alpha)(1+\alpha m)^2 - 6nm(1+\alpha m)(1+4\alpha+6\alpha^2) \\ & - 2\sqrt{6}m(n-1)(1+3\alpha)(1+\alpha m)\sqrt{1+4\alpha+6\alpha^2} \\ & + \sqrt{6}(n-1)m(1+4\alpha+6\alpha^2)^{\frac{3}{2}} = 0. \end{aligned}$$

When $m = 0$ then $\alpha + \frac{1}{3} = 0$ is a simple solution of this equation. Therefore α may be represented as a power series of the form

$$\alpha = -\frac{1}{3} + a_1 m + a_2 m^2 + \dots$$

which converges for sufficiently small values of m . Substituting this expression in the preceding equation and equating to zero the coefficients of the various powers of m it is found that

$$a_1 = \frac{1}{18} - \frac{\sqrt{2}(n-1)}{108},$$

.....

whence

$$\alpha = -\frac{1}{3} + \frac{m}{18} - \frac{\sqrt{2}(n-1)m}{108} + \dots$$

Consequently $-\frac{1}{3}$ is the limit of the value of α as m approaches zero. As m is very small in the lunar theory this value of α is very satisfactory, and particularly so as it changes the trinomial into a binomial.

To illustrate the matter consider a numerical example with a large value of m , say $m = \frac{1}{2}$. Then let the remainder be considered after three terms are taken, or $n-1=3$. It follows that $\alpha = -.318650$, $m' = .594760$, and $(6-4m+m^2) = .235294$. Now it is easily found that

$$[6-4m+m^2]^{-1} = \frac{1}{6}(1 + \frac{2}{3}m + \frac{5}{18}m^2) + R_3(m),$$

and when $\alpha = -\frac{1}{3}$

$$[6-4m+m^2]^{-1} = \frac{1}{6}(1 + \frac{2}{3}m' + \frac{1}{18}m'^2) + R_3(m'),$$

and when $\alpha = -.318650$

$$[6-4m+m^2]^{-1} = \frac{1}{6}(1 + \frac{2}{3}m' + .065345m'^2) + R_3(m').$$

It is found by substituting the numerical values that

$$R_3(m) = .001498,$$

$$R_3'(m') = -.001373,$$

$$R_3(m') = -.001310.$$

These few terms are only enough to illustrate the matter.

The object of a literal expansion of the coefficients is doubtless to enable one to compute them most conveniently for various values of the parameter which may arise in practice. In the lunar theory, where alone there is any demand for extended computations, the parameter m is known with a high degree of accuracy. If it were exactly known there would be no occasion for these expansions for use in the lunar theory, since the power series possess no theoretical advantages over the series of fractions either when retained literally or when reduced to numbers.

If the numerical value is given to m on the start the labor of computing the coefficients is enormously reduced, but they admit of no direct corrections for improved values of the parameter. But if m_0 represents the approximate value of m which is at present known, and if the substitution $m = m_0 + m'$ be made, a series will be obtained in m' which will converge extremely rapidly because of the smallness of this parameter. A very few terms will be sufficient to give the results correctly to many decimals for any value of m' which is apt to appear necessary to use. For practical application in the lunar theory this preserves all the generality attained in using m and secures most of the convenience of using numbers from the start. The unfavorable features are that the numerical coefficients cannot be expressed as ordinary fractions and the trinomial $(6 - 4m + m^2)$ remains a trinomial after the transformation. However, if the substitution

$$m = \frac{m_0 + m'}{1 + \alpha m'}$$

is used, the α may be determined so that the trinomial shall reduce to a binomial after which the expansion may be made with great ease. For values of m far different from m_0 the series in m' converge about as rapidly as those in m .

Chapters XII and XIII, which treat briefly of the principal theories other than those considered earlier in the text and of

the effects of planetary and other disturbing influences, are much to be commended except the paragraph in the center of the last page. In this the astonishing statement is made that, because the moon's orbit shrinks as the eccentricity of the earth's orbit decreases, the moon would fall on the earth if the earth's orbit became circular. Obviously a variable may always decrease without having zero as its limit, and in this case the incorrectness of the conclusion is quite evident.

In recapitulation it may be said that Professor Brown has written a very satisfactory treatise on the lunar theory, the faults being for the most part those which have their origin in the inherent, and up to the present unsolved, difficulties of the subject. There is no better source of information for one who wishes to acquaint himself with this celebrated field of investigation.

F. R. MOULTON.

THE UNIVERSITY OF CHICAGO,
November, 1902.

THE DOCTRINE OF INFINITY.

Die Grundsätze und das Wesen des Unendlichen in der Mathematik und Philosophie. Von Dr. K. GEISSLER. Leipzig, Teubner, 1902. 4to, 417 pp.

THE recent tendency toward critical investigation of the axioms underlying mathematical sciences has led to the necessity of discussing the philosophical basis of the whole mathematical structure, and it is to be hoped that more attention will be given the subject from a purely philosophical standpoint.

The announcement of a book treating the old stumbling block of the infinite from both the mathematical and the philosophical points of view was therefore of the greatest interest, and the character of the book was apparently guaranteed by the excellent reputation of the publishers. But the book—though not lacking in a certain kind of interest—is in several respects most surprising, and, as we shall see, somewhat disappointing.

The first 296 pages are devoted to mathematical investigations, and we shall consider these alone. A great variety of topics are treated, such as the parallel axiom, indeterminate forms, tangents, limits, derivatives, velocity, curvature and many

other problems involving continuity. It is not here proposed to discuss these subjects in detail, but to confine ourselves to one serious matter, which the author regards as fundamental and which certainly affects the whole of his reasoning. This is his introduction of certain (fixed) infinitely great and infinitely small numbers (or segments in geometry).

There are two reasons for considering the subject seriously. In the first place it is common, in certain elementary treatises on the calculus, to introduce such infinitely small quantities and to deal with them under the name of "differentials." From the standpoint of mathematical pedagogy, it is certainly of vital importance to discover to what conclusions the introduction of such numbers leads, both with regard to the correctness of using them at all and with regard to the effect upon the student's conceptions and his mode of thought. Secondly, from the point of view of the foundations of mathematics, it is interesting to investigate the axioms tacitly assumed and to see just what axioms (of the usual set) may be retained and what new axioms need be added in the logical treatment of such infinitely great and infinitely small quantities, capable of assuming fixed constant values.

One of the axioms generally accepted in both arithmetic and geometry is that of Archimedes, which may be stated for arithmetic as follows: *Given any two positive numbers a and b , of which b is the greater, there exists a certain positive integer n , such that $a + a + a + \dots$ (n times) $> b$.*

Passing over the first few pages of the book, which are intended for very elementary students, we find, on page 35, in direct violation of Archimedes's axiom, the introduction of a certain positive ($\neq 0$) number δ , termed an "infinitely small" number, such that

$$\delta + \delta + \delta + \dots (n \text{ times}) < 1$$

for any (finite) integer n . And the "fundamental theorem" is repeatedly stated (*e. g.*, page 141) as follows: "The product of a finite quantity and an infinitesimal is always infinitesimal"; while on page 66 it is stated that "the concept of an infinitesimal does not coincide exactly with zero." *

These statements explicitly deny the axiom of Archimedes, though the author does not mention the fact. Nor will the

* See also p. 46; and p. 90, where an infinitesimal unit is considered.

non-euclidean geometer protest against a geometry in which certain usual axioms are expressly violated, provided the fact of such violation is constantly kept in mind and no appeal whatever is made to ordinary geometry. In particular, one axiom which is at the basis of all considerations of continuity in the ordinary sense having been rejected, we naturally expect a thorough research of the axioms of continuity which are assumed to hold. We shall see that the author has unfortunately not adhered to such a programme.

No result of the book can then be accepted, even for a non-euclidean, non-archimedean, geometry. But it is interesting to note that some of the results obtained are such as actually do hold in the non-archimedean geometry actually constructed by Hilbert, in his *Grundlagen der Geometrie*.

If we consider as our realm of numbers all the rational functions of a certain parameter t , and define "equality," "greater than," and "less than" by the rule $f(t) >, =, < g(t)$ according as $f(t) - g(t) >, =, < 0$ for all sufficiently large positive values of t , then all the axioms of arithmetic (see Hilbert, page 26), except the axiom of Archimedes, are satisfied.*

We find, for instance, in the book in hand, on page 62, that "a ratio of two infinite [or infinitesimal] numbers of the same class is a finite number, * * * and a product of two infinite numbers of the same class is not finite, but is infinite [or infinitesimal] of another class." Arrangement into classes might be made by the rule that numbers of any one class satisfy, among themselves, the axiom of Archimedes. It is then readily seen that the above statement holds true in Hilbert's non-archimedean geometry. The two numbers

$$\frac{a_m t^m + a_{m-1} t^{m-1} + \dots + a_0}{b_n t^n + b_{n-1} t^{n-1} + \dots + b_0} \quad \text{and} \quad \frac{c_r t^r + c_{r-1} t^{r-1} + \dots + c_0}{d_s t^s + d_{s-1} t^{s-1} + \dots + d_0}$$

are of the same class if $m - n = r - s$. Their quotient is of the class $(m - n) - (r - s) = 0$, and is hence "finite," i. e., in the same class with 1; while their product is of class $(m - n) + (r - s)$ which is neither zero nor equal to $(m - n)$ unless $m - n = 0$. This latter exceptional case corresponds to the statement (page 62): "The class of finite numbers is that in which the product and the ratio of any two numbers of the class again belongs to the class" — which holds for positive

*The form is slightly different from Hilbert's.

numbers. We are also informed that the choice of which class is to be considered finite is not at our disposal.

On page 177 the author says "the motion of sliding" — (presumably continuous motion in euclidean space) "is to be thought of not in general, but only for a certain class of segments at a time." The notion expressed practically amounts to saying that the complete (non-archimedean) geometry assumed does not permit of continuous motion in the ordinary sense. The author is at great pains elsewhere* to explain away the difficulty that a passage from "finite" to "infinite" [or to "infinitesimal"] numbers, for instance, can be affected *neither* by a sudden spring, nor yet *without* such a spring. That he has violated a fundamental axiom of continuity, and hence cannot expect his geometry to be continuous in the ordinary sense, is not made clear.

Again, on page 194, for example, the author is surprised to find himself in difficulties with such words as "continuous," "hole-less," etc., as applied to space. And he declares that a segment (in geometry) or a number (in arithmetic), when added to a (greater) second, "does not alter the value of the second except when it is of the same class." And on the next page (195): "for this reason and only for this reason we shall be correct if we consider in any discussion the numbers (or segments) of any one class." But this is returning to archimedean geometry, and relinquishing our previous violation of the archimedean axiom. And, unfortunately for this assertion, numbers and segments of different classes *are* treated in the same discussion constantly throughout the book (see pages 28, 29, 51, 59, 77, etc.).

Finally we find, on page 196, "the limit of a (finite) variable x is a region (precisely expressed, a region of finiteness); and this region is infinitely small, but otherwise arbitrary."

Such are, indeed, in part, the necessary conclusions of a non-archimedean geometry. But it should be carefully noted that such results have nothing to do with euclidean geometry, *i. e.*, with space as *ordinarily* conceived and treated. It is the author's lack of appreciation of this fact, and his constant appeal to the intuition (and to even much less strict metaphysical persuasion) which robs the book of any value whatever, even as a discussion of non-archimedean geometry. Even the archimedean axiom itself seems to be used implicitly at

* See *c. g.*, p. 10.

points, in assuming that the geometry is continuous in the ordinary sense, but the arguments are so intermingled with intuitional conclusions that it is wholly impossible to decipher just what axioms the author is implicitly assuming. It is scarcely necessary to state that the discussions of limits, infinite series, the parallel axiom,* derivates, etc., are riddled with errors; and the minor matters treated are no less free from gross mistakes. That Leibniz's ideas of infinity are the nearest approach to the ideas of this book, as the author frequently asserts, is in itself a sufficient condemnation.

The moral of the book concerns the thoughtless introduction of "differentials" in our ordinary books on elementary calculus.† A "differential," treated as an "infinitesimal," or "infinitely small quantity," is precisely a non-archimedean number; and no reasoning can be applied to a system of numbers in which such "infinitesimals" occur without an investigation of the extraordinary properties of such a non-archimedean (non-euclidean) arithmetic (or geometry).

On the other hand, if differentials are to be introduced in the calculus, we may do so (in two dimensions) by considering the perfectly finite quantities

$$dx = \Delta x, \quad dy = \frac{dy}{dx} \cdot \Delta x, \quad d_2y = \frac{d^2y}{dx^2} \cdot \Delta x^2,$$

etc., where Δx is a certain fixed finite increment of x chosen at pleasure. Such a treatment can be made perfectly rigorous, and the advantages (if any such exist) of the differential notation can be preserved, while our geometry and arithmetic remain euclidean and archimedean. But the differentials here considered are perfectly finite quantities, with no taint of peculiarity or obscurity.

The errors in conceptions and in modes of thought, induced in a student by the use of differentials in a loose sense, are well illustrated by the nature of the conceptions of this author. And on serious consideration the treatment of calculus by the use of "infinitely small" differentials must be discarded ‡ by

* A "proof" of this axiom is given on p. 26. The author states that previous proofs are lacking in rigor!

† And to a minor extent, the ordinary calculus treatment of indeterminate forms, and of infinite series.

‡ This statement of course presupposes that the elementary course on the calculus is to be based on euclidean geometry, and is to use intuitive proofs of theorems involving continuity and limits, to some extent.

any sincere writer as totally illogical in itself, and baneful in its effects upon the student.

While this is but one phase of the book in hand, it is the most important phase, to which all else is made subordinate. We will then note but one other point: namely, that the author protests against a violation of the axiom that two points always determine a straight line (page 44), but assumes that "a whole is greater than a part" even when the "whole" in question is infinite (page 19). In both particulars the modern tendency is certainly quite the opposite.

The philosophical portion of the book will not be criticised here, but it is scarcely felt that a philosopher would care to be sponsor for the statements made — certainly not for the style in which they are presented. The philosophical views of mathematical thought, and in particular of infinity and infinitesimals, must surely take into account, however, the positive results now in the possession of mathematicians regarding the effect of the violation of the archimedean axiom upon our system of axioms and upon our conceptions of space.

E. R. HEDRICK.

SHEFFIELD SCIENTIFIC SCHOOL,
November, 1902.

SOME RECENT GERMAN TEXT-BOOKS IN GEOMETRY.

Lehrbuch der Elementar-Geometrie. Erster Theil, Planimetrie.

Von Dr. E. GLINZER. Siebente Auflage. Dresden, Kühnmann, 1899. 4to, 120 pp.

Grundriss der Geometrie. I. Planimetrie. Von J. H. KÜHL.

Zweite Auflage. Dresden, Kühnmann, 1900. 4to, 95 pp.

Lehrbuch der Stereometrie. Von Dr. P. SAUERBECK. Stuttgart, Kröner, 1900. 4to, 291 pp.

Die Elemente der analytischen Geometrie. Zweiter Theil: Die

analytische Geometrie des Raumes. Von Dr. F. RUDIO.

Dritte, verbesserte Auflage. Leipzig, Teubner, 1901. 4to, vi + 186 pp.

THE authors of the first two texts are connected with the institution in Hamburg known as the Allgemeine Gewerbeschule und Baugewerkschule. One might expect accordingly

a general disregard of anything approaching euclidean rigor in their treatment of the subject. In fact, no precise statement of axioms or definitions is anywhere to be found in either book. The parallel axiom, as such, does not appear at all. Dr. Glinzer gives Euclid's form, viz.: If two lines are cut by a third so that the sum of the interior angles on one side is less than a straight angle, these lines will, if sufficiently produced, meet on that side — as a *proposition*, the proof following it involving comparison of infinite areas. The trivial nature of this proof is, however, emphasized by the setting in small type. The other author assures the pupil that the proof of the proposition: If two lines are cut by a third under equal alternate interior angles, the lines are parallel, does not necessarily imply the truth of the converse, but the correctness of the latter is easily established graphically.

Irrational numbers receive scant attention, the propositions involving incommensurable lengths being regarded as relatively unimportant. The word "limit" is not mentioned at all, but the pupil is told off-hand to apply the formulæ for regular polygons to the case when the number of sides become infinite.

All of this goes to show that, in this German industrial school, the facts of geometry are to be learned at any cost, the development of the student's logical faculties being a matter of secondary importance. A geometric sense must be cultivated and the feeling instilled in the learner that drawing a figure is often half the battle in proving a theorem in geometry. To this end, much attention is paid to constructive geometry, a feature worthy of imitation in American texts.

In extent the subject matter is much the same as the traditional plane geometry. Type and figures are clear, but hardly up to the standard of our best American books.

In the preface to Dr. Sauerbeck's volume the author registers his opinion that "the first object of solid geometry is the cultivation of space intuition." With this view the reviewer is heartily in agreement, and he believes that the usual American text-book contributes very little to this end. For the stress is felt by the student to be laid on the *form* of proof, and not on the *content* of the proposition. After a strenuous course in plane geometry, it may be assumed that euclidean form and rigor in demonstration have been acquired, and in the remainder of the course, emphasis should be put not on logical develop-

ment and sequence of propositions, but on the facts of solid geometry and spatial relations. The traditional course in American colleges supplementing the usual requirement in plane geometry for this reason results in a paltry addition to the sum of the student's knowledge of geometry. How much greater would be the gain in power if an amount of ground could be covered similar to that presented in Dr. Sauerbeck's text! For in pages 1-59 the author discusses the usual theorems involving metrical relations of planes and lines, but in addition such of the elements of projective geometry as naturally arise in the consideration of central projection, the principle of duality, etc. Next follows a section on crystallography, well done but of doubtful value. Section V, pages 84-133, treating of the polyhedral angle and the sphere, is quite complete, including radical planes, axes, centers of similitude, etc. Following this (pages 133-211) surfaces of revolution are discussed, and in this section the subject of conic sections finds consideration, together with stereographic projection and applications. The closing pages treat of mensuration. In these the theorem of Cavalieri is assigned its proper place beside the method of exhaustion. One wonders why this extremely simple, evident and powerful theorem has won no place in American texts. The theorems of Guldin are proven, but seem out of place, at least in their general form. The very general class of solids known as Simpson's solids furnish an excellent example in cubature, and give, as analogous formulæ in the plane, the well-known Simpson's rule for quadrature. A commendable feature of the book is the large number of exercises, solved and unsolved.

To repeat, the position of the author seems to the reviewer a very strong one, and it is to be hoped that the coming years will show a departure from the present prevalent practice of teaching solid geometry in our colleges after the manner of the secondary schools.

Dr. Rudio's book, with its companion volume on the plane, is deservedly popular in Germany. Evidence of great care and thoroughness in writing is everywhere abundant throughout the volume, and the result is a most excellent text. The principle of projection is adopted most wisely as fundamental, and the fixing of the algebraic sign of lengths, areas and volumes carefully treated. The author has kept strictly within the

limits set by the title, the book closing with a discussion of the elementary properties of quadrics, such as tangent planes, diameters, rectilinear generators, etc. The collection of exercises is large, and figures and type are clear and attractive.

PERCEY F. SMITH.

SHEFFIELD SCIENTIFIC SCHOOL,
December, 1902.

NOTES.

THE opening (January) number of volume 4 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Orthocentric properties of the plane n -line," by FRANK MORLEY; "Definitions of a field by independent postulates," by L. E. DICKSON; "Definitions of a linear associative algebra by independent postulates," by L. E. DICKSON; "Two definitions of a commutative group by sets of independent postulates," by E. V. HUNTINGTON; "Definitions of a field (Körper) by sets of independent postulates," by E. V. HUNTINGTON; "On the invariants of differential forms of degree higher than two," by C. N. HASKINS; "Ueber die Reducibilität der Gruppen linearer homogener Substitutionen," by ALFRED LOEWY; "The quartic curve as related to conics," by A. B. COBLE; "The cogredient and digredient theories of multiple binary forms," by EDWARD KASNER; "On the envelopes of the axes of a system of conics passing through three points," by R. E. ALLARDICE; "A Jordan curve of positive area," by W. F. OSGOOD.

THE January number (volume 25, number 1) of the *American Journal of Mathematics* contains: "The parametric representation of the tetrahedroid surface," by D. N. LEHMER; "On ternary monomial substitution groups of finite order with determinant ± 1 ," by E. B. SKINNER; "On forms of unicursal sextic scrolls," and "On forms of sextic scrolls of genus one," by VIRGIL SNYDER; "Note on symmetric functions," by E. D. ROE, Jr.

THE number contains a portrait of Professor L. CREMONA.

THE lectures delivered by Professor OSKAR BOLZA before the colloquium of the AMERICAN MATHEMATICAL SOCIETY at Ithaca, N. Y., in August, 1901, will be published early in the

spring as one of the supplementary volumes of the Decennial publications of the University of Chicago, under the title : "Lectures on the calculus of variations."

THE committee on definitions of college entrance requirements in mathematics, appointed by the AMERICAN MATHEMATICAL SOCIETY at the summer meeting held at Evanston, September, 1902, presented to the Council, on December 29, a tentative report, in the form of a memorandum. In this memorandum are defined the requirements in algebra, plane geometry, solid geometry, higher algebra, and plane trigonometry. Algebra covers the elementary subject through the progressions, with the note that a college finding it impracticable to meet this normal requirement is advised to demand algebra through quadratics, and that a college preferring to hold two examinations is advised to make the division immediately before quadratic equations. Plane geometry is defined as including the usual theorems and constructions of good standard text-books, the solution of original exercises, applications to problems of mensuration of lines and of plane figures, and to loci problems. Solid geometry is similarly defined. Higher algebra is taken to cover permutations and combinations, applications of the principle of mathematical induction, theory and use of logarithms (not involving infinite series), determinants, elements of the theory of equations (with graphic methods), Horner's method, elements of the theory of complex numbers (with graphic representation, limited to sums and differences). Plane trigonometry is defined to include the six functions as ratios; the proofs of principal formulas, in particular the sine, cosine and tangent of $A \pm B$ and of $2A$, and the product expressions for the sum or the difference of two sines or of two cosines; the properties of logarithms and the use of tables; the solution of triangles, with applications.

THE first winter meeting of the American association for the advancement of science was held at Washington, D. C., during convocation week, December 29, 1902, to January 3, 1903. Among the officers elected for the present year are: President of the Association, CARROLL D. WRIGHT; Vice-President of Section A, A. H. TITTMANN; Secretary of Section A, L. G. WELD. The next meeting of the Association will be held at St. Louis in convocation week, 1903-1904.

At the meeting of the London mathematical society held on December 11, 1902, the following papers were read: By Professor L. E. DICKSON: (1) "The abstract group simply isomorphic with the group of linear fractional transformations in a Galois field;" (2) "Generational relations of an abstract simple group of order 4080;" by Dr. H. F. BAKER: (1) "On the calculation of the finite equations of a continuous group," (2) "On the integration of linear differential equations," (3) "On some cases of matrices with linear invariant factors;" by Professor M. J. M. HILL: "The continuation of the power series for $\arcsin x$;" by Mr. E. T. WHITTAKER: "The functions associated with the parabolic cylinder in harmonic analysis;" by Mr. H. M. MACDONALD: "Some applications of Fourier's theorem;" by Rev. F. H. JACKSON: "Series connected with the enumeration of partitions;" by Mr. W. H. YOUNG: "Sets of intervals, part ii: overlapping intervals;" By Mr. G. H. HARDY: "On the expression of the double zeta and gamma functions in terms of elliptic functions;" by Mr. J. H. GRACE: "Perpetuants (second paper)."

OXFORD UNIVERSITY.—The following courses in mathematics are announced for Hilary term, 1903: By Professor W. ESSON: Comparison of analytic and synthetic methods in the geometry of conics, two hours; Synthetic geometry of cubics, one hour.—By Professor E. B. ELLIOTT: Elements of elliptic functions, two hours.—By Professor A. E. H. LOVE: Attractions and electrostatics, two hours; Theory of potential, one hour.—By Mr. A. L. DIXON: Calculus of finite differences, one hour.—By Mr. J. E. CAMPBELL: Algebra of quantics, one hour.—By Mr. P. J. KIRKBY: Higher plane curves, two hours.—By Mr. C. H. SAMPSON: Solid geometry (continued), two hours.—By Mr. C. E. HASELFOOT: Theory of equations, one hour.—By Mr. J. W. RUSSELL: Pure geometry, two hours.—By Mr. C. LEUDESORF: Geometry (maxima and minima, inversion, etc.), two hours.

At the annual public meeting of the Paris academy of sciences, held on December 22, 1902, the following mathematical prizes were awarded: Grand prize, 3,000 francs, ERNEST VESSIOT; very honorable mention, JEAN LE ROUX; Bordin prize, 3,000 francs, not awarded; very honorable mention, W. DE TANNENBERG; Franceœur prize, 1,000 francs, ÉMILE LEMOINE; Poncelet prize, 2,000 francs, MAURICE D'OCAGNE;

Extraordinary prize of 6,000 francs, divided between M. ROMAZOTTI and M. DRIENCOURT.

THE International congress of historical sciences, which was projected to be held at Rome in April, 1902, but was afterward postponed, will meet in that city on April 2-9, 1903. Eight sections have been arranged, of which the last is devoted to the history of mathematical, physical, natural and medical sciences. The invitations to participate in section VIII are issued by a committee of which Professor Gino Loria of Genoa is the representative of the interests of pure mathematics. Information concerning the congress may be obtained by addressing the Segretariato generale del congresso internazionale di scienze storiche, Collegio Romano, via del Collegio Romano, 26, Rome.

THE press of Girardi and Audebert, of Dole, France, has recently issued a circular of information, in English, for the benefit of American and other foreign students who contemplate attending the French universities. The circular covers particularly the questions of matriculation and degrees, but it also gives valuable information concerning the various university centers. The Report, just issued, of the U. S. Commissioner of Education, 1901, volume 1, page 1113, gives similar information.

THE first annual *Sitzungsbericht* of the Berlin mathematical society, has been reprinted from the *Archiv der Mathematik und Physik*, and published separately. Presumably it is intended to issue these proceedings annually in this form.

THE following academic appointments and promotions have recently been announced: DR. ALBERT BENTELI has been appointed professor of geometry in the University of Bern. DR. A. SUCHARDA has been made professor of mathematics in the Bohemian Technical high school at Brünn. DR. V. VARICAK has been made professor of mathematics in the University of Agram. M. GUITTON, of the lycée at Caen, has been appointed to a lectureship in mathematics in the faculty of sciences in that city; M. ZORETTI to an instructorship in mathematics in the Ecole normale supérieure, Paris.

AMONG recently elected foreign members of the Royal Society are Dr. G. W. HILL and Professor GASTON DARBOUX. The Society has awarded its medal to Professor HORACE LAMB for his investigations in mathematical physics.

PROFESSOR LUIGI CREMONA, of Rome, has been elected a foreign member of the American academy of arts and sciences, of Boston.

PROFESSOR C. SEGRE, of Turin, has been elected an honorary fellow of the Cambridge philosophical society.

MR. G. T. WALKER, fellow and lecturer of Trinity College, Cambridge, has been appointed director-general of the Indian meteorological bureau.

LIEUTENANT MAX GENTY, known through numerous mathematical contributions, died at Toulon on October 24, aged 35 years.

THE name of the eminent Russian mathematician whose death was noticed on page 223 of the January number of the BULLETIN should have been printed Professor NIKOLAUS BUGAIEV.

RECENT catalogue of second-hand mathematical works: Fr. Strobel, Jena; general science, including mathematics, 1012 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABEL (N. H.). Festschrift ved Hundredaarsjubilaet for Niels Henrik Abels Födsel. Christiania, 1902. 4to. 357 pp. 2 portraits, 6 facsimiles. M. 14.50

ABHANDLUNGEN zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, begründet von M. Cantor. Heft 14. Inhalt: A. A. Björno, Studien über Menelaos' Sphärik; Beiträge zur Geschichte der Sphärik und Trigonometrie der Griechen; H. Suter, Nachträge und Berichtigungen zu "Die Mathematiker und Astronomen der Araber und ihre Werke"; K. Bopp, Antoine Arnauld, der grosse Arnauld, als Mathematiker. Leipzig, Teubner, 1902. 8vo. 8 + 338 pp.

AHL (F.). Untersuchungen über geodätische Linien. Kiel, 1901. 8vo. 50 pp. M. 1.50

BJÖRNO (A. A.). See ABHANDLUNGEN.

BOPP (K.). See ABHANDLUNGEN.

GEIGENMÜLLER (R.). Leitfaden und Aufgabensammlung zur höheren Mathematik; für technische Lehranstalten und den Selbstunterricht bearbeitet. Vol. I: Die analytische Geometrie der Ebene und die algebraische Analysis. 6te Auflage. Mittweida, Polytechnische Buchhandlung, 1902. 8vo. 7 + 302 pp. Cloth. M. 6.50

- GERNET (N.). Untersuchung zur Variationsrechnung. Ueber eine neue Methode in der Variationsrechnung. (Diss.) Göttingen, 1902. 8vo. 75 pp.
- GREINER (A.). Ueber orthogonale Invarianten der Kurven dritter Ordnung mit unendlich fernem Doppelpunkt und ihre geometrische Bedeutung. (Diss.) Jena, 1902. 8vo. 41 pp.
- HEDRICK (E. R.). The English and French translations of Hilbert's Grundlagen der Geometrie. 8vo. 1902. (*Bulletin of the American Mathematical Society* (2) 9, pp. 158-165.)
- HETTNER (G.). See WEIERSTRASS (K.).
- KNOBLAUCH (J.). See WEIERSTRASS (K.).
- KOENIGSBERGER (L.). Hermann von Helmholtz. Vol. I. Braunschweig, Vieweg, 1902. 8vo. 12 + 375 pp., 3 portraits. M. 8.00
- KUTTA (W.). Beiträge zur näherungsweise Integration totaler Differentialgleichungen. (Diss.) München, 1901. 8vo. 19 pp.
- LOREY (W.). Ueber das geometrische Mittel, insbesondere über eine dadurch bewirkte Annäherung kubischer Irrationalitäten. (Diss.) Halle, 1902. 4to. 27 pp.
- LUDWIG (W.). Die Horopterkurve, mit einer Einleitung in die Theorie der kubischen Raumkurve. Halle, Schilling, 1902. 8vo. 36 pp. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von M. Schilling in Halle a. S. Neue Folge, No. 3.) M. 1.00
- NERNST (W.) und SCHÖNFLIES (A.). Einführung in die mathematische Behandlung der Naturwissenschaften. Kurzgefasstes Lehrbuch der Differential- und Integralrechnung mit besonderer Berücksichtigung der Chemie. 3te Auflage. München und Berlin, Oldenbourg, 1902. 8vo. M. 10.00
- PASCAL (E.). I gruppi continui di trasformazioni; parte generale della teoria. Milano, Hoepli, 1903. 16mo. 11 + 358 pp. (Manuali Hoepli.)
- . Lezioni di calcolo infinitesimale. Parte II: Calcolo integrale. 2a edizione, completamente riveduta. Milano, Hoepli, 1902. 16mo. 8 + 329 pp. (Manuali Hoepli.)
- PERRY (N.). Das Problem der konformen Abbildung für eine spezielle Kurve von der Ordnung $3n$. (Diss.) München, 1901. 8vo. 34 pp.
- ROUGUET (V.). Etude géométrique des surfaces dont les lignes de courbure d'un système sont planes et égales. Marseille, 1902. 4to. 43 pp.
- SCHÖNFLIES (A.). See NERNST (W.).
- STEPHANSEN (M. A. E.). Ueber partielle Differentialgleichungen 4ter Ordnung, die ein intermediäres Integral besitzen. Kristiania, Cammermeyer, 1902. 8vo. 80 pp. M. 2.80
- SUTER (H.). See ABHANDLUNGEN.
- WEIERSTRASS (K.). Mathematische Werke. Vol. IV: Vorlesungen über die Theorie der Abelschen Transcendenten. Bearbeitet von G. Hettner und J. Knoblauch. Berlin, Mayer & Müller, 1902. 4to. 14 + 632 pp. M. 40.00
- WÖRNER (K.). Ueber eine besondere Gattung von Gruppen. Freiberg, 1902. 8vo. 36 pp. M. 1.20

II. ELEMENTARY MATHEMATICS.

- AGOLINI UGOLINI (G.). La geometrica od esatta divisione sessagesimale della circonferenza, ossia la scoperta dell'angolo di un grado; geometrica inscrizione nel circolo dei poligoni regolari di 9 e 45 lati e loro conseguenti; dimostrazione matematica. Sanseverino-Marche, Bellabarba, 1902. 8vo. 15 pp.
- ASHTON (C. H.) and MARSH (W. R.). Plane and spherical trigonometry; an elementary text-book. New York, Scribner, 1902. 8vo. 10 + 157 pp. Cloth.
- BARDEY (E.). Anleitung zur Auflösung eingekleideter algebraischer Aufgaben. 2te, völlig umgearbeitete Auflage von F. Pietzker. Leipzig, Teubner, 1903. 8vo. 8 + 159 pp. Cloth. M. 2.60
- BÉCHÉ (A.). See DUSSAUX (E.).
- BORTEVICH (I.). Elementary trigonometry, with exercises and problems. 3d edition. St. Petersburg, 1902. 4to. 94 pp. (Russian.) M. 3.00
- BIGGS (W.). Matriculation advanced algebra and geometry; required for Syllabus in advanced mathematics of London University matriculation examination. London, Clive, 1902. 12mo. 332 pp. Cloth. (University tutorial series.) 4s. 6d.
- BRUCE (W. H.). Some noteworthy properties of the triangle and its circles. Boston, Heath, 1902. 12mo. 28 pp. \$0.10
- COMBEROUSSE (C. DE). Curso de matemáticas. Vol. I, parte 1: Aritmética, traducida del francés por E. Prado, para uso de los alumnos del Colegio militar de Chapultepec. México, Gallegos, 1902. 8vo. 242 pp.
- CRAWLEY (E. S.). A short course in plane and spherical trigonometry. Published by the author. 8vo. 116 + 27 pp. Cloth. \$1.00
- DUSSAUX (E.) et BÉCHÉ (A.). Troisième année de géométrie dans l'enseignement primaire supérieur. Paris, Colin, 1902. 16mo. 285 pp. (Collection J. Boitel.) Fr. 2.50
- ERNST (C.) und STOLTE (L.). Lehrbuch der Geometrie zum Gebrauche an Gymnasien, Realschulen und anderen höheren Lehranstalten. Teil I: Planimetrie, nebst einer Sammlung von Aufgaben. 4te Auflage. Strassburg, Strassburger Druckerei und Verlagsanstalt, 1903. 8vo. 109 pp. Boards. M. 1.50
- F. (F.). Géométrie (cours moyen). Paris, Poussielgue, [1902]. 16mo. 191 pp. (Collection d'ouvrages classiques rédigés en cours gradués.)
- FORMULAIRE de mathématiques spéciales, à l'usage des candidats aux écoles polytechniques, centrales, des mines, des ponts et chaussées, etc., etc. (Algèbre, géométrie analytique à deux et trois dimensions, trigonométrie, mécanique.) Paris, Rudeval, 1902. 18mo. 75 pp. Fr. 1.25
- GAMBIOLI (D.). Breve sommario della storia delle matematiche, con 2 appendici sui matematici italiani e sui 3 celebri problemi geometrici dell'antichità. Bologna, 1902. 12mo. 239 pp. M. 2.50

JACQUET (E.) et LACLEF (A.). Cours de géométrie théorique et pratique, avec de nombreux exercices, problèmes, applications, etc., à l'usage des écoles normales d'instituteurs et d'institutrices, des écoles primaires supérieures, des écoles professionnelles et des candidats au brevet supérieur. Paris, Nathan, 1902. 16mo. 488 pp.

JIMÉNEZ RUEDA (C.). Programa de geometría métrica. Madrid, Suárez, 1902. 8vo. 20 pp. (Universidad central.) Fr. 1.00

KOPPE und DIEKMANN. Geometrie zum Gebrauche an höheren Unterrichtsanstalten. Teil 1 und 2 der Planimetrie, Stereometrie und Trigonometrie. Ausgabe für Reallehranstalten. Teil 1. (21ste Auflage.) 5te Auflage der neuen Bearbeitung von J. Diekmann. 4 + 248 pp. Teil 2. (18te Auflage.) 2te Auflage der neuen Bearbeitung von J. Diekmann. 4 + 268 pp. Essen, Baedeker, 1903. 8vo. Cloth. M. 4.80

LACLEF (A.). See JACQUET (E.).

LASALA Y AMIZAR. Elementos de matemáticas. Vol. I: Aritmética, 240 pp. Vol. II: Algebra, 304 pp. México, Herrero, 1902. 8vo. \$1.50

MACÉ DE LÉPINAY (A.). See VACQUANT (C.).

MALAGODI (A.). Nozioni d'algebra elementare, con numerosi esercizi e problemi ad uso delle scuole tecniche e dei ginnasi. 3a edizione migliorata. Mirandola, Cagarelli, 1902. 8vo. 77 pp. Fr. 0.60

MARSH (W. R.). See ASHTON (C. H.).

MARTINI ZUCCAGNI (A.). Trattato di algebra elementare ad uso degli istituti tecnici. Livorno, Giusti, 1902. 8vo. 10 + 362 pp. Fr. 2.80

MAYER (J. E.). Das mathematische Pensum des Primaners. Ein Hilfsbuch für den Primaner humanistischer und realistischer Gymnasien. (In 10-12 Heften.) Heft I: Progressionen, Zinsseszins- und Rentenrechnung. Freiburg i. B., Lorenz, 1902. 8vo. 52 pp. M. 1.00

MÜLLER (E.). Lehr- und Uebungsbuch der ebenen Geometrie mit besonderer Berücksichtigung des Zusammenhanges zwischen Lehrsatz und Konstruktionsaufgabe; für Gymnasien und Realschulen. Berlin, Winckelmann, 1903. 8vo. 6 + 172 pp. M. 1.80

NASSÒ (M.). Aritmetica generale ed algebra ad uso dei licei secondo il programma governativo del 24 ottobre 1900, con copiose note storiche, molti consigli pratici per indirizzare l'alunno alla risoluzione degli esercizi, più di 2200 esercizi e problemi graduati da risolvere e circa 400 esercizi e problemi minutamente risolti. 2a edizione, interamente rifatta. Torino, Tipografia Salesiana, 1902. 8vo. 504 pp.

NIEWENGLOWSKI (B.). Cours d'algebre, à l'usage des élèves de la classe de mathématiques spéciales et des candidates à l'Ecole polytechnique. 5e édition, entièrement refondue. Vol. II. Paris, Colin, 1902. 8vo. 492 pp. Fr. 8.00

PAVLOV (N.). Methodology of the elements of arithmetic, for the guidance of the teacher. Kazan, 1902. 8vo. 270 pp. (Russian.) M. 2.00

PIETZKER (F.). See BARDEY (E.).

PRADO (E.). See COMBEROUSSE (C. DE.).

PURYEAR (C.). See TAYLOR (T. U.).

SHULIN (G. I.). Spherical geometry and trigonometry. 2d edition. St. Petersburg, 1902. 8vo. 151 pp. (Russian.)

SPIEKER (T.). Lehrbuch der ebenen Geometrie mit Übungsaufgaben, für höhere Lehranstalten. Ausgabe C. Abgekürzte Kurse. 2te Auflage. Potsdam, Stein, 1903. 8vo. 4 + 205 pp. M. 2.00

STOLTE (L.). See ERNST (C.).

TAYLOR (T. U.) and PURYEAR (C.). The elements of plane and spherical trigonometry. Boston, Ginn, 1902. 8vo. 160 + 67 pp. Cloth. \$1.25

TESTI (G. M.). Sulla ricerca di una soluzione di una equazione di primo grado a due incognite. Livorno, Giusti, 1902. 8vo. 4 pp.

—. Sulla risoluzione dei sistemi di disequaglianze. Livorno, Giusti, 1902. 8vo. 7 pp.

VACQUANT (C.) et MACÉ DE LÉPINAY (A.). Éléments de géométrie, à l'usage des classes de sciences. 2e édition. Premier cycle (classes de cinquième, de quatrième et de troisième). Paris, Masson, 1903. 16mo. 435 pp.

III. APPLIED MATHEMATICS.

ALEXANDER (T.) and THOMPSON (A. W.). Elementary applied mechanics; with numerous diagrams, and a series of graduated examples carefully worked out. London and New York, Macmillan, 1902. 8vo. 20 + 575 pp. Cloth. \$5.25

AUERBACH (F.). Die Grundbegriffe der modernen Naturlehre. Leipzig, 1902. 8vo. 4 + 156 pp. M. 1.00

BARCHANEK (K.). Lehr- und Übungsbuch der darstellenden Geometrie. Leipzig, 1902. 8vo. 8 + 374 pp. Cloth. M. 5.00

BIGELOW (F. H.). Studies on the statics and kinematics of the atmosphere in the United States. Reprints from the *Monthly Weather Review*, January to July, 1902. Washington, Weather Bureau, 1902. 4to. 4 + 62 pp.

BISKE (F.). Versuch einer Anwendung hydrodynamischer Untersuchungen auf die Protuberanzen der Sonne. (Diss.) Berlin, 1901. 8vo. 37 pp.

BJERKNES (V.). Vorlesungen über hydrodynamische Fernkräfte nach C. A. Bjerknes' Theorie. Band II: Versuch über hydrodynamische Fernkräfte; Analogie hydrodynamischer Erscheinungen mit elektrischen oder magnetischen. Leipzig, Barth, 1902. 8vo. 16 + 314 pp. M. 10.00

BURN (J.) and BROWN (E. H.). Elements of finite differences, also solutions to questions set for part I of the examinations of the Institute of Actuaries. London, 1902. 8vo. Cloth. 7s. 6d.

DREYFUS (L.). See POINCARÉ (H.).

- ENGESTRÖM (F. VON). Lifförsäkring. Del I och II: Beräkning af premier och matematiska värden; försäkringsvilkor och premier. Stockholm, 1902. 8vo. 124 + 158 pp. M. 4.80
- HECHT (K.). Lehrbuch der reinen und angewandten Mechanik für Maschinen- und Bautechniker. Vol. III: Die graphischen Methoden; mit 225 Beispielen und einem Anhang. Dresden, Kühnmann, 1903. 8vo. 7 + 600 pp. M. 12.00
- KRAGH (O.). Studier over Pendulbevaegelsen. Kjöbenhavn, 1902. Cloth. M. 4.00
- LAURENT (H.). Traité de perspective, à l'usage des peintres et des dessinateurs de profession ou des personnes qui désirent se faciliter l'étude du dessin. Paris, Schmid, [1902]. 8vo. 76 pp.
- LE CONTE (J. N.). An elementary treatise on the mechanics of machinery; with special reference to the mechanics of the steam-engine. New York, Macmillan, 1902. 12mo. 10 + 311 pp., 15 plates. Cloth. \$2.25
- PARENT (L.). Perspective élémentaire. Paris, Schmid, [1902]. 8vo. 95 pp., 36 plates.
- POINCARÉ (H.). Cours de physique mathématique. Figures d'équilibre d'une masse fluide; leçons professées à la Sorbonne en 1900, rédigées par L. Dreyfus. Paris, Naud, 1902. 8vo. 215 pp.
- SCHLOTKE (J.). Lehrbuch der darstellenden Geometrie. Teil I: Spezielle darstellende Geometrie. 5te (Titel-) Auflage. 4 + 167 pp. Teil II: Schatten- und Beleuchtungslehre. 3te Auflage. 60 pp. Teil III: Perspektive. 2te Auflage. 5 + 133 pp. Dresden, Kühnmann, 1902. 8vo. M. 10.00
- . Lehrbuch der graphischen Statik. Zum Gebrauche für mittlere technische Lehranstalten, Bau-, Maschinen- und Gewerbeschulen. 2te verbesserte und vermehrte Auflage. Dresden, Kühnmann, 1902.
- THOMSON (A. W.). See ALEXANDER (T.).
- TRUTAT (E.). Traité général des projections. 2 vols. Paris, 1902. 8vo. Fr. 12.00

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The initiation fees and annual dues of members of the American Mathematical Society are payable to the Treasurer of the American Mathematical Society, 501 West 116th Street, New York.

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501 West 116th Street, New York City.

The following dates have been fixed for the meetings of the Society :

A. M.	S. M.
Sat., Feb. 28, 1903, 11:00	Sat., Oct. 31, 1903, 11:00
" April 25, " "	M.-Tu., Dec. 28-29, " "

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A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

F. N. COLE A. ZIWET D. E. SMITH F. MORLEY

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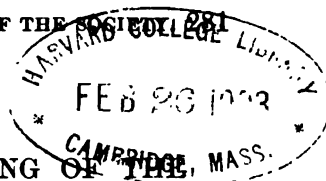
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[March, 1903.] NINTH ANNUAL MEETING OF THE SOCIETY 281



THE NINTH ANNUAL MEETING OF THE
AMERICAN MATHEMATICAL SOCIETY.

THE Ninth Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Monday and Tuesday, December 29-30, 1902. The following sixty-two members were in attendance at the four sessions:

Dr. Grace Andrews, Mr. C. H. Ashton, Professor Joseph Bowden, Professor E. W. Brown, Dr. A. B. Chace, Professor A. S. Chessin, Dr. J. E. Clarke, Dr. A. B. Coble, Dr. A. Cohen, Professor F. N. Cole, Professor L. L. Conant, Mr. D. R. Curtiss, Dr. W. S. Dennett, Dr. Otto Dunkel, Professor T. C. Esty, Dr. William Findlay, Professor H. B. Fine, Professor T. S. Fiske, Dr. G. B. Germann, Dr. G. H. Hallett, Miss Carrie Hammerslough, Professor James Harkness, Dr. H. E. Hawkes, Dr. E. R. Hedrick, Dr. A. A. Himowich, Dr. E. V. Huntington, Dr. S. A. Joffe, Dr. Edward Kasner, Dr. C. J. Keyser, Professor Pomeroy Ladue, Professor P. A. Lambert, Professor J. L. Love, Dr. Emory McClintock, Professor H. P. Manning, Professor W. H. Metzler, Mr. H. B. Mitchell, Professor E. H. Moore, Professor F. Morley, Professor G. D. Olds, Professor W. F. Osgood, Professor Anna H. Palmié, Professor James Pierpont, Professor R. W. Prentiss, Professor M. I. Pupin, Dr. I. E. Rabinovitch, Professor E. D. Roe, Jr., Professor Charlotte A. Scott, Mr. Burke Smith, Professor D. E. Smith, Professor P. F. Smith, Dr. H. F. Stecker, Professor Irving Stringham, Dr. W. M. Strong, Professor H. W. Tyler, Professor E. B. Van Vleck, Dr. Roxana H. Vivian, Professor L. A. Wait, Mr. A. C. Washburne, Professor F. S. Woods, Professor R. S. Woodward, Professor J. W. A. Young, Mr. J. W. Young.

The retiring President of the Society, Professor Eliakim Hastings Moore, presided at the earlier sessions, being relieved by Vice-Presidents Professor F. Morley and W. F. Osgood, and giving way on Tuesday to the President-Elect, Professor Thomas Scott Fiske. The Council announced the election of the following persons to membership in the Society: Dr. A. B. Coble, University of Missouri; Mr. W. A. Cornish, State Normal School, Cortland, N. Y.; Dr. A. G. Hall, University

of Michigan ; Mr. E. L. Hancock, Purdue University ; Professor L. M. Hoskins, Stanford University ; Mr. W. D. A. Westfall, Yale University ; Mr. W. F. White, State Normal School, New Paltz, N. Y. Sixteen applications for admission to the Society were received.

Reports were presented by the Librarian, the Treasurer, and the Auditing Committee. These reports will appear in the Annual Register, now in press. During the past year the membership of the Society has increased from 375 to 399 ; the number of papers presented at the meetings from 133 to 156. The total attendance at all meetings was 266 ; 144 members attended at least one meeting. The library now contains about 1000 bound volumes, and nearly 120 mathematical journals are regularly received in exchange for the *Bulletin* and the *Transactions*. A catalogue of the library will be published in the Annual Register.

Coming at the expiration of the presidential term of office, the meeting was especially marked as the occasion of a presidential address. Under the title : "On the foundations of mathematics," the retiring President, Professor E. H. Moore, urged the desirability of the Society undertaking to exert an effective influence on the teaching of elementary mathematics. A committee of the Council was appointed to consider this question.

At the annual election, which closed on Tuesday morning, the following officers and other members of the Council were chosen :

<i>President,</i>	Professor THOMAS S. FISKE.
<i>Vice-Presidents,</i>	Professor W. F. OSGOOD. Professor ALEXANDER ZIWET.
<i>Secretary,</i>	Professor F. N. COLE.
<i>Treasurer,</i>	DR. W. S. DENNETT.
<i>Librarian,</i>	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor D. E. SMITH.

Members of the Council to serve until December, 1905,

Professor JAMES HARKNESS, Professor IRVING STRINGHAM,
Professor HEINRICH MASCHKE, Professor H. W. TYLER.

The following papers were read at the annual meeting :

(1) Dr. E. V. HUNTINGTON : "A complete set of postulates for the theory of real numbers (second paper)."

(2) Dr. E. V. HUNTINGTON : "On the definitions of the elementary functions by means of definite integrals."

(3) Dr. C. J. KEYSER : "On the axiom of infinity."

(4) Professor G. H. DARWIN : "The approximate determination of the form of Maclaurin's spheroid."

(5) Professor HARRIS HANCOCK : "Remarks on the sufficient conditions in the calculus of variations."

(6) Professor L. E. DICKSON : "The abstract group G simply isomorphic with the alternating group on six letters."

(7) Professor E. H. MOORE : Presidential address : "On the foundations of mathematics."

(8) Professor W. E. TAYLOR : "On the product of an alternant and a symmetric function."

(9) Professor E. D. ROE, JR. : "On the coefficients in the product of an alternant and a symmetric function."

(10) Professor E. D. ROE, JR. : "On the coefficients in the quotient of two alternants" (preliminary communication).

(11) Professor E. O. LOVETT : "A transformation group of $(2n-1)(n-1)$ parameters, and its rôle in the problem of n bodies."

(12) Dr. I. E. RABINOVITCH : "On the solid lunes of conicoids, analogous to the circular lunes of Hippocrates of Chios."

(13) Dr. E. B. WILSON : "The synthetic treatment of conics at the present time."

(14) Dr. A. B. COBLE : "On the invariant theory of the connex $(2, 2)$ of the ternary domain viewed as a connex $(1, 1)$ in a five dimensional space."

(15) Dr. EDWARD KASNER : "The general quadratic systems of conics and quadrics."

(16) Professor W. F. OSGOOD : "On the transformation of the boundary in the case of conformal mapping."

(17) Professor W. F. OSGOOD : "A Jordan curve of positive area."

(18) Mr. J. W. YOUNG : "On the automorphic functions associated with the group of character $[0, 3; 2, 4, \infty]$ " (preliminary report).

(19) Mr. R. W. H. T. HUDSON : "The analytic theory of displacements."

(20) Dr. H. E. HAWKES: "Enumeration of the non-quaternion number systems."

(21) Dr. H. F. STECKER: "On the parameters in certain systems of geodesic lines."

(22) Dr. EDWARD KASNER: "The generalized Beltrami problem concerning geodesic representation."

(23) Professor MAXIME BÔCHER: "Singular points of functions which satisfy partial differential equations of the elliptic type."

(24) Mr. G. D. BIRKHOFF and Mr. H. S. VANDIVER: "General theory of the integral divisors of $a^n - b^n$, and allied cyclotomic forms."

(25) Professor G. A. MILLER: "A new proof of the generalized Wilson theorem."

(26) Professor F. MORLEY: "On the determinant $|(x_i - a_j)^{-2}|$."

Professor Darwin's paper was communicated to the Society through Professor E. W. Brown, Professor Taylor's through Professor Metzler, and Mr. Hudson's through Professor Morley. Mr. Vandiver was introduced by Professor Crawley. In the absence of the authors, Professor Darwin's paper was read by Professor Brown, Professor Taylor's by Professor Roe, Professor Bôcher's by Professor Osgood, and the papers of Professor Hancock, Professor Dickson, Professor Lovett, Dr. Wilson, Mr. Hudson and Professor Miller were read by title.

Dr. Wilson's paper and Professor Osgood's first paper appeared in the February BULLETIN. The papers of Professor Dickson and Mr. Hudson are contained in the present number of the BULLETIN. Abstracts of the other papers are given below. The abstracts are numbered to correspond to the titles in the list above.

1. The two sets of postulates for real number * contained in Dr. Huntington's present paper are more convenient and natural than the set proposed by him at the April meeting (see abstract in BULLETIN, volume 8, page 371). The fundamental concepts involved in the present paper are 1° a *relation*, $\odot$, in which one element may stand to another — $a \odot b$ being read: a "less than" b ; 2° a *rule of combination*, $\oplus$, according to which every two elements a and b of a given assemblage de-

* Cf. D. Hilbert, *Jahresber. d. Deutschen Math.-Vereinigung*, volume 8 (1900), page 180.

termine uniquely an object $a \oplus b$ called their "sum"; and 3° another *rule of combination*, $\odot$, according to which a and b determine uniquely an object $a \odot b$ called their "product." Any assemblage in which $\odot$ and $\oplus$ are so defined as to satisfy the nine postulates of the first set below, or any assemblage in which $\odot$, $\oplus$ and $\ominus$ are so defined as to satisfy the thirteen postulates of the second set, will be equivalent to the system of all real numbers (positive, negative and zero). Each set is "complete"; that is, the postulates of each set are (a) consistent, (b) independent, and (c) sufficient to define essentially a single assemblage.

The first set contains the following postulates :

1. If $a \neq b$ then either $a \odot b$ or $b \odot a$.
2. If $a \odot b$ and $b \odot c$ then $a \odot c$.
3. The relation $c \odot c$ is false for at least one element c .
4. If $a \odot b$ then there is an element x such that $a \odot x$ and $x \odot b$.
5. If S is an infinite sequence of elements a_k such that

$$a_k \odot a_{k+1}, \quad a_k \odot c \quad (k = 1, 2, 3, \dots)$$

(where c is some fixed element), then there is an element A having the following two properties : $a_k \odot A$ whenever a_k belongs to S ; and if $A' \odot A$ then there is an element a_r of S such that $A' \odot a_r$.

6. If a , b and $b \oplus a$ belong to the assemblage, then $a \oplus b = b \oplus a$.

7. If a , b , c , $a \oplus b$, $b \oplus c$ and $a \oplus (b \oplus c)$ belong to the assemblage, then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

8. For every two elements a and b there is an element x such that $a \oplus x = b$.

9. If $x \odot y$ then $a \oplus x \odot a \oplus y$.

The second set contains these nine postulates and also the following four :

10. The product $a \odot b$ always belongs to the assemblage.
11. First distributive law : $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.
12. Second distributive law : $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$.
13. If 0 is an element such that $c \oplus 0 = c$ for every element c , and if $0 \odot a$ and $0 \odot b$, then $0 \odot a \odot b$.

2. The usual plan of defining the elementary functions first for real variables and afterwards, by a process of generalization, for complex variables, involves much unnecessary repetition in

the proofs of many of their properties. In Dr. Huntington's second paper the attempt is made to avoid this repetition by defining the functions once for all for the most general case. In the first place, the fundamental theorems concerning definite integrals along a path in the complex plane (especially Goursat's theorem that the integral of a differentiable function around a closed path is zero) are rigorously developed without presupposing any knowledge of functions of a real variable, except addition, subtraction, multiplication and division. Next, $\log z$ is defined as the integral of dz/z from 1 to z , and the fundamental relation $\log a + \log b = \log(ab)$ obtained by a familiar device.* Then e^z is defined as the inverse of $\log z$, and x^y as $e^{y \log x}$. The existence and properties of these functions follow immediately for the most general case, without separate consideration of irrational and imaginary values. Finally, the integration of such functions as $1/(1+z^2)$ and $1/\sqrt{1-z^2}$ leads very naturally to the definitions of $\tan^{-1}z$ and $\sin^{-1}z$ as abbreviations for certain logarithmic expressions, while $\tan z$ and $\sin z$ appear as the inverses of $\tan^{-1}z$ and $\sin^{-1}z$ respectively.

3. Dr. Keyser's paper undertakes a critical examination (1) of Dedekind's "theorem of complete induction," and (2) of the proofs offered by Dedekind and Bolzano of the so-called existence theorem for infinite assemblages. In respect to (1), it is shown that that theorem, so far from being "the scientific basis of the method of argument called complete induction," can itself be proved by means of mathematical induction provided one recognizes that our faith in the validity of, in the apodictic certainty yielded by, this argument form finds adequate justification in a certain a priori judgment (apparently first formulated as such by Poincaré), here called the axiom of infinity and shown to be a presupposition of all logical discourse. It follows that against all such "proofs" as are referred to under (2) there must lie the charge of circularity. In course of the article several theorems in the doctrine of Ketten are established.

4. The following extract indicates the nature of Professor Darwin's paper, which will appear in the *Transactions*:

* Cf. H. Burkhardt, *Analytische Functionen*, p. 162, and an article by J. W. Bradshaw about to appear in the *Annals of Mathematics*.

"Spherical harmonics render the approximate determination of the figure of a rotating mass of liquid a very simple problem. If ρ be the density, e the ellipticity, and ω the angular velocity of the spheroid, the solution is

$$\frac{\omega^2}{2\pi\rho} = \frac{8}{15}e.$$

This result is only correct as far as the first power of the ellipticity, but M. Poincaré has recently shown (*Philosophical Transactions*, volume 198 A, pages 333–373) how harmonic analysis may be used so as to give results as far as the squares of small quantities; and I have myself used his suggestion for the determination of the stability of the pear-shaped figure of equilibrium (*Philosophical Transactions*, volume in the press).

Both these papers involved the use of ellipsoidal harmonic analysis and it would be rather tiresome to extract the method from the complex analysis in which it is embedded. It therefore seems worth while to treat the well-worn subject of Maclaurin's spheroid as an example of the method in question."

5. Professor Hancock's paper is in summary as follows. In order that the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y') dt,$$

where F is a regular one-valued function of its arguments, may have a maximum or minimum value, it is necessary that the function F have the two properties: 1° It must be homogeneous of the first degree in x' and y' ; 2° The partial derivatives $\partial F/\partial x'$ and $\partial F/\partial y'$ must be continuous, even if there are sudden changes in x' and y' . We have further the four conditions: 1° The differential equation $G=0$ must be satisfied for every point of the curve within the interval $t_0 \dots t_1$; 2° The function F_1 must retain the same sign within the interval, and cannot become zero or infinite within this interval; 3° There cannot be conjugate points within this interval; 4° The function $E(x, y, p, q, \bar{p}, \bar{q})$ must always have the same sign within the interval $t_0 \dots t_1$. The conditions 2° and 3° are necessary to establish the condition 4°.

8. The object of Professor Taylor's paper is, in extension of a problem started by Muir, to determine independently the co-

efficient of any alternant occurring in the product of a symmetric function of weight w and the alternant $|0123 \dots n-1|$. Proceeding from the standpoint of Muir, the symmetric function is regarded as being first expressed in the usual manner as a sum of products of elementary symmetric functions of weight w with certain coefficients. The problem then becomes, to find the coefficient of any alternant in the product of the single alternant and a product of elementary symmetric functions of weight w . A general reduction formula for such a coefficient is found. In particular the coefficient of the alternants of the type $|0123 \dots r_1 s_1 \dots r_2 s_2 \dots r_3 s_3|$, where $s_1 - r_1 = 2$, $s_2 - r_2 = 2$, $s_3 - r_3 = q$, in the product $|012 \dots n-1| \sum a_1 a_2 \dots a_{n-1} (\sum a_i)^i$ is found to be

$$\frac{i(i-1) \dots (i-s_1+2)}{|s_1-1|} \cdot \frac{(s_1-2)(s_1-3) \dots (s_1+r_3-s_3-2)}{|s_3-r_3-1|}.$$

If tables of coefficients for weight w be made, others of the same type for weight $w + j_h$ can be constructed by means of the preceding reduction formula. Tables of this kind from weight one to weight seven were given. The coefficient of any alternant in the product of the original symmetric function and $|0123 \dots (n-1)|$ can then be obtained by multiplying the coefficient of the table of weight w by the coefficient of the elementary symmetric function product in the symmetric function table, and by taking the sum of all such products.

9. In Professor Roe's paper, which is an extension of a previous one by the author, the direct product of the symmetric function $\sum \alpha_1^{p_1} \alpha_2^{p_2} \dots \alpha_m^{p_m}$ and the alternant $|\lambda_1 \lambda_2 \dots \lambda_m|$ is considered. The coefficient of any alternant $|\kappa_1 \kappa_2 \dots \kappa_m|$ in the product is first shown to be equal to the coefficient of $\alpha_{p_1} \alpha_{p_2} \dots \alpha_{p_m}$ in the determinant

$$\begin{vmatrix} a_{\kappa_1-\lambda_1} & a_{\kappa_2-\lambda_1} & \dots & a_{\kappa_m-\lambda_1} \\ a_{\kappa_1-\lambda_2} & a_{\kappa_2-\lambda_2} & \dots & a_{\kappa_m-\lambda_2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{\kappa_1-\lambda_m} & a_{\kappa_2-\lambda_m} & \dots & a_{\kappa_m-\lambda_m} \end{vmatrix}$$

and then a general recurrence formula for the coefficient is established. A method of calculating it by means of bringing about the identity of two complexes is also explained. It is

next shown how a product table for all symmetric functions of the preceding type of weight w and the alternant $|012 \dots m - 1|$ can be so constructed by placing conjugate partitions in a vertical, and conjugate alternants in a horizontal line, equidistant from each end, that all the coefficients lie in a triangle whose hypotenuse consists of ones, and the sum of whose columns is always zero, excepting the last one which is unity. For this table general formulas are given for the coefficients. Moreover it is shown that the whole table is absolutely invariant and still gives the complete products if $|012 \dots m + r - 1|$ is used as multiplier. It is shown how the coefficients of a quotient table are certain determinants taken from the product table, and that Dr. Taylor's *product* table is the author's *quotient* table.

As it was the author's original object to obtain a method of computing symmetric functions, it is finally shown that the table is a table for all the symmetric functions of weight w written in the vertical line.

10. In his second paper Professor Roe stated for the quotient table of alternants mentioned in his first paper, eight properties which he has already observed.

11. In Professor Lovett's paper attention is called to a group of $(2n - 1)(n - 1)$ transformations, in as many variables, which the author encountered in a study of the problem of n bodies, the law of force being an arbitrary function of the distance. The invariants of this group are functions of the integrals of the problem of n bodies which are independent of the law of force. This interpretation appears by choosing as the variables certain quadratic functions of the coördinates and velocities of the bodies of the system.

The paper itself is concerned with the case where n is arbitrary, but the group for the particular case $n = 4$ may be presented in this brief abstract as a typical one; its twenty-one infinitesimal transformations may be written as follows :

$$X_{1,123}f \equiv 2s_{1,q_1} + r_{1,s_1} + s_{21,q_{12}} + s_{31,q_{13}} + r_{21,s_{12}} + r_{31,s_{13}},$$

$$X_{2,123}f \equiv 2s_{1,r_1} + q_{1,s_1} + s_{12,r_{12}} + s_{13,r_{13}} + q_{21,s_{21}} + q_{31,s_{31}},$$

$$X_{3,123}f \equiv 2s_{21,r_1} + 2s_{12,r_2} + q_{12,q_1} + q_{12,q_2} + q_{1,r_{12}} + q_{2,r_{12}} + s_{13,r_{33}} \\ + s_{23,r_{21}} + q_{1,s_{12}} + q_{2,s_{21}} + q_{32,s_{31}} + q_{31,s_{32}},$$

$$X_{4,123}f \equiv 2q_{1,q_1} - 2r_{1,r_1} + q_{12,q_{12}} + q_{13,q_{13}} - r_{12,r_{12}} - r_{13,r_{13}} + s_{12,s_{12}} \\ + s_{13,s_{13}} - s_{21,s_{21}} - s_{31,s_{31}},$$

$$X_{5,123}f \equiv 2q_{12,q_1} - 2r_{12,r_2} + s_{21,s_1} - s_{21,s_2} + q_{2,q_{12}} + q_{32,q_{13}} - r_{1,r_{12}} \\ - r_{31,r_{23}} + s_{2,s_{12}} - s_{1,s_{12}} + s_{23,s_{13}} - s_{31,s_{23}},$$

$$X_{6,123}f \equiv 2q_{12,q_2} - 2r_{12,r_1} + s_{12,s_2} - s_{12,s_1} + q_{1,q_{12}} + q_{31,q_{23}} - r_{2,r_{12}} \\ - r_{32,r_{21}} + s_{1,s_{21}} - s_{2,s_{21}} + s_{13,s_{23}} - s_{32,s_{21}},$$

$$X_{7,123}f \equiv 2s_{12,q_1} + 2s_{21,q_2} + r_{12,s_1} + r_{12,s_2} + s_{1,q_{12}} + s_{2,q_{12}} + s_{31,q_{23}} \\ + s_{32,q_{21}} + r_{2,s_{12}} + r_{1,s_{21}} + r_{23,s_{13}} + r_{31,s_{23}},$$

$$X_{i,231}f, \quad X_{i,312}f, \quad (i = 1, 2, 3, \dots, 7),$$

where $a_{i,b_j} \equiv a_i \partial f / \partial b_j$; $q_y = q_y$; $r_y = r_y$; $s_y = s_y$.

12. Dr. Rabinovitch considered the surface of revolution $f(\sqrt{x^2 + y^2}, z) = 0$, referred to rectangular coördinates, the axis of z being the axis of revolution. The meridian planes (1) $y = 0$ and (2) $y = mx$, will cut each parallel circle $z = k$, $x^2 + y^2 = \rho^2$ into four segments, supplementary two by two. Describing upon these in each parallel plane the lunes of Hippocrates, we obtain four solid lunes, whose volume is proved to be equal to the volume of the common portion of the two intersecting cylinders, whose directrices are the meridian curves of the given surface of revolution, situated respectively in the planes (1) and (2), and whose corresponding generatrices are respectively parallel to the intersections of each parallel plane $z = k$ with (2) and (1). In the particular case of surfaces of revolution of the second degree, the volume of these lunes, between two parallel planes, does not involve any other irrationality than that which may be involved in the arbitrary distance $z_1 - z_0$ between the bounding planes. Thus, the expression for the volume of a solid, bounded by portions of a surface of revolution is free from the irrationality π .

In particular, the total volume of the lunes upon a sphere of radius a equals

$$\frac{1}{2} \cdot 8 \sin \theta \int_0^a (a^2 - z^2) dz = \frac{8}{3} a^3 \sin \theta,$$

θ being the angle of inclination of the meridional planes (1) and (2). For an hyperboloid of revolution of one sheet, where

the meridian curve is an equilateral hyperbola, the volume of the lunes between the planes $z = z_0$ and $z = z_1$ equals

$$4 \sin \theta \int_{z_0}^{z_1} (a^2 + z^2) dz.$$

The analogy between these solid lunes and the circular lunes of Hippocrates is evident. All other ellipsoidal and hyperboloidal lunes are easily reduced to these. Paraboloidal lunes are also considered.

14. The connex (2, 2) of the ternary domain $a^2 u_a^2 = 0$ coordinates to a line conic u_y^2 , the line conic $a_y^2 u_a^2$; to a point conic c_x^2 , the point conic $c_x^2 a_x^2$. If point and line conics be considered as flat spaces and points respectively in an S_5 , the above coördination is expressed analytically as a connex (1, 1) in S_5 . To a property of the connex (1, 1) in S_5 invariant under the projective group G_{35} of S_5 will correspond a property of the connex (2, 2) in S_2 invariant under the group G_8 of the plane which appears in S_5 as a subgroup of G_{35} .

Hence the invariant theory of the ternary connex (2, 2) breaks into two parts, one part invariant only under the projective group of the plane, the other invariant under a larger group. This second part may be very advantageously treated by reference to an S_5 . Clebsch's (Lindemann) treatment of the connex (1, 1) in S_5 may be followed out for the connex (1, 1) in S_5 and the results obtained readily translated to apply to the connex (2, 2) in S_2 .

Some facts as to the first part also are readily proven by considering the projective group of the plane as a subgroup of G_{35} in S_5 which leaves invariant a certain 4-spread and 2-spread. *E. g.*, The necessary and sufficient condition that the two quartics $a_x^2 b_x^2 (a\beta y)^2$ and $u_a^2 u_\beta^2 (abv)^2$ may be put respectively in the forms

$$\sum_1^6 (c_x^2)^2 \quad \text{and} \quad \sum_1^6 (u_y^2)^2,$$

where the six conics c_x^2 touch the line v and the six conics u_y^2 go through the point y , is that $(abv)^2 (a\beta y)^2 = 0$.

15. Veronese, Segre and others have considered what may be termed the geometry of conics in which the conic defined

by six homogeneous coördinates $r_{11} : r_{22} : r_{33} : r_{23} : r_{13} : r_{12}$ is regarded as the element. Dr. Kasner's first paper is a contribution to this aspect of geometry. A linear equation in the coördinates $r_{i,k}$ defines a linear system of conics whose theory is well known. The general quadratic equation defines the quadratic system Q which is studied in detail by the author. It consists of ∞^4 conics, two of which pass through an arbitrary set of four points. The discussion of the degenerate conics contained in the system leads first to a correspondence defined by a quadri-quadric form f in two sets of line coördinates, and second to a curve of fourth class f , which is the envelope of the double lines contained in Q . The quartic covariant curve f is entirely general; to any quartic there correspond ∞^5 quadratic systems Q . The conics common to Q and one, two, or three linear systems are discussed. Finally analogous results are obtained for a quadratic system of quadric surfaces.

17. Peano has shown that a continuous curve

$$x = f(t), \quad y = \phi(t), \quad 0 \leq t \leq 1,$$

may fill a two dimensional region, in particular, the interior of a square. Peano's curve, however, has multiple points. If such points are excluded, then a curve defined by the above equations is called a Jordan curve. The question presents itself: May the external area of a Jordan curve have a positive value? This question is answered in the affirmative in Professor Osgood's second paper by means of an example. The paper has been published in the January number of the *Transactions*.

18. Mr. Young's paper is a continuation of the paper presented by him to the Society at its last summer meeting. He considers the theta-fuchsian functions of Poincaré associated with the group Γ of character $\{0, 3; 2, 4, \infty\}$ (Fricke-Klein notation), which has for fundamental circle the real axis. Every theta-fuchsian function of Γ of order $4\kappa + \alpha$ which does not become infinite at any point in the positive half-plane is the product of κ such theta-fuchsian functions of order 4 and one of order α , and has just κ movable zeros in a fundamental region C of Γ , unless $\alpha = 1$; in the latter case, it is the product of $\kappa - 1$ functions of order 4 and two of orders 2 and 3 respectively, and has just $\kappa - 1$ movable zeros in C . Theta-

fuchsian functions of any order exist with the proper number of arbitrarily assigned movable zeros. It is shown, further, how every such function can be rationally expressed in terms of two functions θ and ϕ , which are simple rational functions of two particular hyperelliptic theta constants. The functions θ and ϕ each vanish in a single point of C , and are such that θ^2 and $\theta\phi$ are theta-fuchsian functions of Γ of orders 2 and 3 respectively.

20. Scheffers has classified all hypercomplex number systems into quaternion or non-quaternion groups. Dr. Hawkes's paper gave a method for enumerating all types of non-quaternion systems of order n from types of order $n - 1$. These types follow the usual principles of classification in being inequivalent, irreducible, non-reciprocal, and with moduli. The method deduced is founded on certain theorems of Benjamin Peirce as extended by Dr. Hawkes in the *Transactions*, volume 3.

21. Dr. Stecker's paper justifies certain forms of equations of doubly infinite systems of geodesic lines used in a paper presented a year ago, where such systems were projected conformally into systems of conics.

22. The problem discussed in Dr. Kasner's second paper may be stated as follows: Given a doubly infinite system of plane curves $F(x, y, \lambda, \mu) = 0$, to investigate the existence and properties of those surfaces which can be represented point by point upon the plane in such a manner that the geodesics of the surface are represented by the assigned curves. Beltrami solved the problem for the straight lines, the corresponding surfaces being those of constant curvature. Dr. Stecker (volumes 2 and 3 of the *Transactions*) has considered the case of a linear system

$$\lambda\phi(x, y) + \mu\psi(x, y) + \chi(x, y) = 0,$$

imposing the additional requirement that the representation should be conformal. This case, however, may be reduced to Beltrami's; the surfaces are simply those of constant curvature, and the linear system, for conformal representation, must be conformally equivalent to a linear system of circles.

In the case of an arbitrary system $F = 0$ no solution in general exists. The necessary and sufficient conditions depend

upon the consistency of a set of four partial differential equations in three unknown functions. The discussion of the number of solutions is carried out by means of Dini's theorem, the surfaces of Liouville type playing an exceptional rôle. Among the important results obtained are the two following: The only surfaces whose geodesics are capable of representation by circles are those of constant curvature. The only surfaces whose geodesics can be conformally represented by curves of the form $y = a_0 x^n + a_1 x^{n-1} + \dots + a_n$ are the developable surfaces and those whose linear element may be written $ds^2 = v(du^2 + dv^2)$. The paper will appear in the *Transactions*.

23. Many properties of Laplace's equation in two dimensions can be deduced from the theory of functions of a complex variable. In this way the two following theorems may be easily proved, in each of which we suppose the function $u(x, y)$ to be harmonic throughout the neighborhood of a point P : 1° If u becomes infinite for *no* method of approaching P , it will, if properly defined at P , be harmonic there. 2° If u becomes infinite for *every* method of approaching P , it has the form $C \log r + v$ where C is a constant, r the distance from P , and v a function which is harmonic at P . According to a remark of Professor Osgood, the first of these theorems is an immediate consequence of the second. In the present paper Professor Bôcher shows how the second theorem (and therefore also the first) can be extended to cases to which the original method of proof was in no wise applicable — on the one hand to Laplace's equation in three dimensions, and on the other hand to the general linear partial differential equation of the second order of the elliptic type in two independent variables and with analytic coefficients. The method of proof consists in an application of Green's theorem.

24. In the paper of Mr. Birkhoff and Mr. Vandiver the complement of two well-known arithmetical congruence theories is considered. The two theories (given in most text-books) are: 1° Theory of power residues; 2° Theory of the solution of binomial congruences. But the relation $A^n - 1 \equiv 0 \pmod{p}$, or more generally $a^n - b^n \equiv 0 \pmod{p}$, where A, a, b and p are integers, may be considered from another standpoint; given a, b and n , we may ask the question: What are the properties of the divisors of $a^n - b^n$, or will certain classes of

divisors always exist? The theory arising from efforts to solve these problems may be regarded as a complement to the two theories just mentioned, and forms the subject of the present discussion.

A primitive divisor of $a^n - b^n$ is defined as a divisor which is prime to all the factors of the form $V_k = a^k - b^k$, k being less than n . Then, using several theorems due to Euler and Sylvester (which are also proved), the following theorem (which we believe new) is arrived at: If $n > 2$, then every integral form $a^n - b^n$ possesses at least one primitive divisor, with the single exception $2^8 - 1$.

The paper concludes with applications of the theory of cyclotomic divisors to the "instantaneous" demonstration of the two theorems: 1° If n is any integer, there is an indefinite number of primes of the form $1 \pmod{n}$. 2° The equation $(x^p - 1)/(x - 1) = 0$ is irreducible if p is prime.

25. The main steps in the proof given in Professor Miller's paper are as follows: It is first observed that the continued product of all the operators of an abelian group G is the identity whenever G contains more than one operator of order 2. If G contains only one operator S of order two, then S is always the continued product of all the operators of G . The group of isomorphisms I of a cyclic group C must contain more than one operator of order 2 unless the order of C is one of the three numbers $4, p^2, 2p^2$, where p represents any odd prime. Without employing the theory of congruences it is proved that in these three special cases I contains just one operator of order 2. The generalized Wilson's theorem follows directly from these facts. The paper has been offered to the *Annals of Mathematics* for publication.

26. In Professor Morley's paper the factors of the determinant $|(x_i - a_j)^{-2}|$ are found by a process which appears to be novel; and thereby the geometric theorem is proved that, given a rational norm curve R_n through $n + 1$ points in S_n , a quadric apolar to the $n + 1$ points cuts R_n in points which form a system conjugate with the $n + 1$ spaces S_{n-1} determined by the $n + 1$ points.

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THE DECEMBER MEETING OF THE SAN FRANCISCO SECTION.

THE second regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held on Saturday, December 20, 1902, at the University of California. The following sixteen members were present :

Professor R. E. Allardice, Dr. E. M. Blake, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor R. L. Green, Professor L. M. Hoskins, Dr. D. A. Lehmer, Dr. J. H. McDonald, Professor G. A. Miller, Dr. A. C. Moreno, Dr. C. A. Noble, Dr. T. M. Putnam, Professor Irving Stringham, Dr. S. D. Townley, Mr. A. W. Whitney, Professor E. J. Wilczynski.

A morning and an afternoon session were held, Professor Stringham acting as chairman at both sessions. The officers elected at the May meeting were reelected for one year, and the by-law relating to the name of the section was amended by replacing "Pacific" by "San Francisco."

The following papers were read at this meeting :

(1) Professor R. E. ALLARDICE : "On a system of similar conics through three points and its transformation group."

(2) Professor H. F. BLICHFELDT : "On a property of conic sections."

(3) Professor L. E. DICKSON : "Generational relations for the abstract group G simply isomorphic with the linear fractional group in the Galois field of order p^n ."

(4) Professor L. M. HOSKINS : "A simple method of determining the free nutation of a yielding spheroid."

(5) Dr. D. N. LEHMER : "On the parametric representation of the tetrahedroid surface."

(6) Professor A. O. LEUSCHNER : "Elimination of aberration and parallax in calculating preliminary orbits."

(7) Mr. W. A. MANNING : "The positive primitive substitution groups of class $2p$, p being any prime."

(8) Professor G. A. MILLER : "On the holomorph of a cyclic group."

(9) Dr. C. A. NOBLE : "A problem in relative minima."

(10) Dr. S. D. TOWNLEY: "The probability of collisions amongst the stars."

(11) Professor E. J. WILCZYNSKI: "On a certain congruence associated with a given ruled surface."

Professor Dickson's paper was read by the secretary. In the absence of Professor Leuschner his paper was read by title. All the other papers were read by their authors and most of them aroused considerable discussion. Professor Blichfeldt's paper appears in the present number of the BULLETIN. Abstracts of the other papers are given below.

1. On considering the loci and envelopes connected with a system of similar conics through three fixed points, Professor Allardice found that a number of curves occur which are intimately associated with the circle. In looking for an explanation of this fact, he found that such a system of conics possesses a group of transformations of the following character; Each transformation is equivalent to a quadric inversion ($x = 1/x$, etc.), followed by a rotation about the circumcenter, and then by a second quadric inversion. Any transformation of the group transforms any conic through the given points into a similar conic through the same points; but the transformations are themselves independent of the eccentricity of the conic. The path curves of the group were determined.

3. Professor Moore has given (see Dickson's Linear Groups, page 300) a set of generational relations for the group G considered by Professor Dickson. The object of the present paper is to give a simpler set of such relations, the proof being confined as yet to the twenty-two cases in which $p^n < 49$. Two cases $p = 2$ and $p > 2$ are essentially different.

If $n = 1$ and p is an odd prime, a complete set of generational relations for G is

$$T^2 = S^p = (ST)^3 = I, \quad (S^{\tau} T S^{2\tau} T)^2 = I$$

where $\tau = 3, \dots, \frac{1}{2}(p-1)$, but only one of the positive residues modulo p of $\tau, 2/\tau, -2/\tau$ is retained.

For $p = 3$ or 5 , the only relations are

$$T^2 = I, \quad S^p = I, \quad (ST)^3 = I.$$

Additional relations are :

$$p = 7, \quad (S^3TS^3T)^2 = I.$$

$$p = 11, \quad (S^3TS^8T)^2 = I, \quad (S^4TS^6T)^2 = I.$$

$$p = 13, \quad (S^3TS^5T)^2 = I, \quad (S^4TS^7T)^2 = I.$$

$$p = 17, \quad (S^3TS^{12}T)^2 = I, \quad (S^4TS^9T)^2 = I, \\ (S^6TS^6T)^2 = I, \quad (S^7TS^{10}T)^2 = I.$$

$$p = 19, \quad (S^3TS^7T)^2 = I, \quad (S^4TS^{10}T)^2 = I, \\ (S^5TS^8T)^2 = I, \quad (S^6TS^{13}T)^2 = I.$$

$$p = 23, \quad (S^3TS^{16}T)^2 = I, \quad (S^4TS^{12}T)^2 = I, \\ (S^5TS^5T)^2 = I, \quad (S^6TS^6T)^2 = I, \quad (S^9TS^{13}T)^2 = I.$$

$$p = 29, \quad (S^3TS^{20}T)^2 = I, \quad (S^4TS^{15}T)^2 = I, \quad (S^5TS^{12}T)^2 = I, \\ (S^6TS^{10}T)^2 = I, \quad (S^7TS^{21}T)^2 = I, \quad (S^{11}TS^{16}T)^2 = I.$$

For any n and any odd prime p , the set is

$$T^2 = I, \quad (S_1T)^3 = I, \quad S_0 = I, \quad S_\lambda S_\mu = S_{\lambda+\mu} \quad (\lambda, \mu \text{ any marks}), \\ (S_\tau TS_{2/\tau} T)^2 = I \quad (\tau \text{ any mark} \neq 0).$$

These relations are somewhat redundant. For $p^n = 9$, the group of order 360 is generated by T , S_1 and S_j , subject only to the relations

$$T^2 = S_1^3 = S_j^3 = I, \quad S_1S_j = S_jS_1, \quad (S_1T)^3 = I, \quad (S_jTS_jT)^2 = I, \\ (S_jS_1TS_j^{-1}S_1T)^2 = I.$$

Its isomorphism with the alternating group on six letters follows from the correspondence $T \sim (12)(34)$, $S_1 \sim (123)$, $S_j \sim (456)$.

For $p = 2$, the set of relations is

$$A^{2^n+1} = I, \quad B^2 = I, \quad (AB)^3 = I \quad (BA^rBA^s)^2 = I,$$

$r = 1, 2, \dots, 2^n$, the value of S being uniquely determined modulo $2^n + 1$ by r . Among the pairs r, s always occur 1, 2 ;

2, 1; $2^n, 2^n - 1$; $2^n - 1, 2^n$; and the resulting relations follow from those giving the periods of A , B , and AB . For $n = 1$ or 2, there are no further relations. For $n = 3$, the relations are

$$A^9 = I, B^2 = I, (AB)^3 = I, (BA^3BA^5)^2 = I.$$

For $n = 4$, the relations are

$$A^{17} = I, B^2 = I, (AB)^3 = I, (BA^3BA^7)^2 = I, \\ (BA^4BA^{12})^2 = I, (BA^6BA^9)^2 = I.$$

For $n = 5$, we may restrict r, s to the values 3, 25; 4, 10; 5, 17; 6, 15; 7, 11; 9, 14; 12, 20. The papers giving the proofs have been presented to the American and London Mathematical Societies.

4. The paper by Professor Hoskins discusses the free rotation of a nearly spherical body whose elastic properties and density have spherical symmetry about the center of mass. By taking as axes of reference the lines which at any instant would coincide with the principal axes of inertia if the strain due to rotation were annulled, the equations of angular momentum take simple forms. If A_0, B_0, C_0 are the principal moments of inertia for the unstrained body, and if I is the increment produced in the moment of inertia about the instantaneous axis by the centrifugal forces due to the actual angular velocity, the equations of motion are the same as for a rigid body whose principal moments of inertia are $A_0 + I, B_0 + I$ and $C_0 + I$. Since the differences of these quantities are the same as those of A_0, B_0 and C_0 , and since I is a small fraction of each of the actual moments of inertia (though not necessarily of their differences), the period of the free nutation is to a close approximation the same as for a rigid body whose principal moments are A_0, B_0 and C_0 . The result is general in that it is not restricted to the case in which the axis of rotation deviates but slightly from its mean position, nor to the case in which two of the principal moments of inertia are equal.

5. Dr. Lehmer employed the σ functions of Weierstrass in the parametric representation of the tetrahedroid surface and showed the arrangement of the singular points, the singular

planes, and their interrelations. The 120 lines joining the 16 singular points were shown to be the 120 lines in which the 16 angular planes intersect. Each of these lines meets 12 others in 6 points in involution. The paper has appeared in the January number (volume 25, number 1) of the *American Journal of Mathematics*.

6. Professor Leuschner's paper begins with a discussion of the expediency of taking into account the corrections for aberration and parallax. The author confirms his former conclusions (Beiträge zur Kometenbahnbestimmung) that in certain cases the resulting orbit will be considerably improved by their elimination. He then reviews the existing methods of accomplishing this elimination and points out that no method of eliminating parallax exists which is applicable in all cases, as even the "reduction to the locus fictus" fails for small geocentric latitudes. The author then proposes some very simple and general formulas for eliminating the parallax by means of corrections to the rectangular heliocentric coördinates of the sun. The corrections depend upon the parallax factors and naturally are independent of the geocentric distance of the body. The paper will be published in the *Bulletin of the Lick Observatory*.

7. Mr. Manning obtains the following results. All positive primitive groups of class $2p$, p a prime > 3 , contain a primitive subgroup of degree $2p + 1$. None exist unless $2p + 1$ is a prime or a power of 3. For every prime number > 3 satisfying one or the other of these conditions there are just two groups, one of degree $2p + 1$ and the other of degree $2p + 2$. When $2p + 1$ is a prime the group of that degree is semimeta-cyclic and the group of degree $2p + 2$ is the modular group. When $2p + 1$ is a power of 3 the first group is a subgroup of the holomorph of the abelian group of order 3^p and type $(1, 1, \dots)$. The group of degree $2p + 2 = 3^p + 1$ is the Mathieu group of order $(3^p - 1)3^p(3^p + 1)/2$. The paper will be offered to the *Transactions* for publication.

8. It is known that the holomorph k of a cyclic group G is a complete group and that its commutator subgroup is G whenever the order g of G is odd. When g is even and greater than 2 the commutator subgroup of K is the subgroup of G whose order is $g/2$, and K is never complete. The main object

of Professor Miller's paper is to determine additional useful properties of K . In particular the order of every operator of K is determined and also the degree of each substitution when K is represented as a transitive substitution group of degree g . It is pointed out that the properties of the group of isomorphisms of G furnish a very simple proof of Fermat's theorem. Some of the properties of the group of isomorphisms of K when g is even are developed. This paper will be offered to the *Transactions*.

9. On a former occasion (dissertation, Göttingen, 1901), Dr. Noble proved the existence of a function $y = y(x)$ which would render the integral

$$\int_{x_0}^{x_1} f(y, y'; x) dx$$

a minimum, provided the integrand satisfied the conditions

$$|f(y, y'; x)| \geq k > 0, \quad \left| f(y, y'; x) \frac{dx}{dy} \right| \geq k > 0$$

(k arbitrary constant).

The method employed in this existence proof was new—suggested by Hilbert in his Göttingen lectures—in that the minimizing curve was progressively constructed from cluster points, and finally identified with the solution of the Lagrange differential equation by means of the appropriate Weierstrass E -function. The object of the present paper is to extend this existence proof to the case of an integral

$$\int_{x_0}^{x_1} f(y, y', y''; x) dx,$$

under analogous restrictions for the integrand $f(y, y', y''; x)$. Such an integral arises in those problems of relative minima in which it is desired to minimize an integral

$$\int_{x_0}^{x_1} F(y, y', z, z'; x) dx,$$

by functions $y = y(x)$, $z = z(x)$ which satisfy the conditioning differential equation $z = \phi(y, y'; x)$. The method is the same

as in the existence proof for the simpler case cited above, the theory of the Weierstrass E -function, as developed by Zermelo, serving to identify the resulting minimal curve with the Lagrange solution.

10. A collision between two stars has often been advanced as an explanation of the phenomena of new stars. Dr. Townley has computed the probability of a collision, assuming a uniform distribution of stars in a finite universe of radius of 400 light years. The method adopted for the solution of the problem brings in two series each containing over two million terms. Only roughly approximate values of the sums of these series have thus far been obtained, but it is hoped to obtain more accurate values of them before long. The computation gives an exceedingly small fraction for the probability of a collision; so small indeed as to make the hypothesis of a collision between two stars no longer tenable as a rational explanation of the phenomena of new stars. No reasonable modification of the assumptions upon which the computations are based will produce a probability sufficiently great to explain the observed number of new stars.

11. In a paper recently published in the *Transactions*, Professor Wilczynski has studied a congruence made up of all of the generators of the first kind on the hyperboloids osculating a given ruled surface. This congruence, which there made its appearance in connection with the theory of covariants of a system of linear differential equations, is considered more fully in the present paper. It is found that its focal surfaces are the flecnodal surfaces of the original surface. The theory of the developable surfaces contained in the congruence, and the theory of certain other characteristic families of ruled surfaces of the congruence are fully developed. Certain theorems, whose formulation in the previous paper was somewhat inadequate, are completed. Incidentally a simple interpretation is found for the canonical form previously adopted by the author for a system of differential equations of the kind here considered. The paper will be offered to the *Transactions* for publication.

G. A. MILLER,
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THE ABSTRACT GROUP G SIMPLY ISOMORPHIC WITH THE ALTERNATING GROUP ON SIX LETTERS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, December 29, 1902.)

1. A SLIGHT correction of a theorem due to De Séguier * leads to the result that G is generated by three operators a, b, c , subject only to the relations

$$(1) \quad a^2 = I, \quad b^4 = I, \quad c^3 = I, \quad (ac)^3 = I,$$

$$(2) \quad (ab^{-1}ab)^3 = I, \quad (ab^{-2}ab^2)^2 = I,$$

$$(3) \quad (cb^{-1}ab)^2 = I, \quad (cb^{-2}ab^2)^2 = I.$$

But these generators are not independent, since

$$(4) \quad a = cb^{-1}cbc.$$

A simple verification of (4) results from the correspondence

$$a \sim (12)(34), \quad b \sim (12)(3456), \quad c \sim (123)$$

between the generators of the simply isomorphic groups.

It is shown in this section that G is generated by the two operators b and c , subject to the complete set of generational relations

$$(5) \quad b^4 = I, \quad c^3 = I, \quad (b^{-1}cbc^{-1})^2 = I, \quad (b^2c)^4 = I.$$

These relations follow from (1), (2), (3); for, by the above correspondence, $b^{-1}cbc^{-1} \sim (14)(23)$, $b^2c \sim (1235)(46)$.

If a be defined by (4), relations (1), (2), (3) follow from (5).

$$a^2 = cb^{-1}cbc^{-1}b^{-1}cbc = c(b^{-1}cbc^{-1})^2c^{-1} = I,$$

$$(ac)^3 = cb^{-1}c^3bc^{-1} = I.$$

**Journal de Math.*, 1902, p. 262. For $y=2, \dots, n-3$ in his formula (6), should stand $y=1, \dots, n-4$.

As an auxiliary result, we note that

$$(6) \quad c(bcb^{-1}cb) = (bcb^{-1}cb)c.$$

The condition (6) may be given the successive forms

$$\begin{aligned} cbc b^{-1} \cdot cbc^{-1} b^{-1} \cdot c^{-1} bc^{-1} b^{-1} &= I, \\ cbc b^{-1} \cdot bcb^{-1} c^{-1} \cdot c^{-1} bc^{-1} b^{-1} &= (cbc^{-1} b^{-1})^2 = I \quad [\text{by } (5_3)]. \end{aligned}$$

Since (6) may be written $cba c^{-1} = ba$, we have

$$(7) \quad c \cdot ba = ba \cdot c.$$

In view of (6) and (5₃), we get

$$(8) \quad (cb^{-1}cb)^3 = cb^{-1} \cdot bcb^{-1}cb \cdot c^2 b^{-1}cb = I.$$

To verify (3₁) we note that, by (8),

$$cb^{-1}ab = cb^{-1} \cdot cb^{-1}cb \cdot b = cb^{-2}c^{-1}bc^{-1}b^{-1}c^{-1}b^2.$$

In the indicated square of the latter, we replace b^2cb^{-2} by $c^{-1}b^2c^{-1}b^2c^{-1}$, in view of (5₄), and transform by cb^2 , and get

$$\begin{aligned} &c^{-1} \cdot bc^{-1}b^{-1}c \cdot b^2c^{-1}b^2 \cdot cbc^{-1}b^{-1} \cdot c^{-1}b^2cb^2 \\ &= c^{-1} \cdot c^{-1}bcb^{-1} \cdot b^2c^{-1}b^2 \cdot bcb^{-1}c^{-1} \cdot c^{-1}b^2cb^2 \quad [\text{by } (5_3)] \\ &= cb \cdot cbc^{-1}b^{-1} \cdot cb^{-1}cb^2cb^2 \\ &= cb \cdot bcb^{-1}c^{-1} \cdot cb^{-1}cb^2cb^2 = (cb^2)^4 = I \quad [\text{by } (5_4)]. \end{aligned}$$

To verify (3₂), we transform by b^2c^{-1} and get

$$\begin{aligned} &cb^2cb^2 \cdot cb^{-1}cb \cdot cb^2cb^2 \cdot cb^{-1}cb \\ &= b^2c^{-1}b^2c^{-1} \cdot cb^{-1}cb \cdot b^2c^{-1}b^2c^{-1} \cdot cb^{-1}cb \\ &= b^{-1}(b^{-1}c^{-1}bc)^2b = I. \end{aligned}$$

To verify (2₁), we note that

$$\begin{aligned} ab^{-1}ab &= cb^{-1}cb \cdot cb^{-1}cb^{-1}cbcb \\ &= cb^{-1}cb(cb^{-1}cb^{-1}cbcb)^{-1} = cb^{-2}c^{-1}bc^{-1}bc^{-1}. \end{aligned}$$

Cubing the inverse and transforming by c , we get (8).

To verify (2₂), we note that

$$\begin{aligned} ab^{-2}ab^2 &= ac^{-1} \cdot cb^{-2}ab^2 = ac^{-1} \cdot b^{-2}a^{-1}b^2c^{-1} \\ &= cb^{-1}cb^{-1}c^{-1}b^{-1}c^{-1}bc^{-1}b^2c^{-1}. \end{aligned}$$

Transforming its square by cb^2 , we get

$$\begin{aligned} & bcb^{-1}c^{-1} \cdot b^{-1}c^{-1}bc^{-1} \cdot bcb^{-1}c^{-1} \cdot b^{-1}c^{-1}bc^{-1} \\ &= cbc^{-1}b^{-1} \cdot b^{-1}c^{-1}bc^{-1} \cdot cbc^{-1}b^{-1} \cdot b^{-1}c^{-1}bc^{-1} \\ &= cbc^{-1}b^{-2}c^{-1}b^2c^{-1}b^{-2}c^{-1}bc^{-1}. \end{aligned}$$

Transforming by cb and taking the inverse, we get $(b^2c)^4 = I$.

2. In a paper entitled "The abstract group simply isomorphic with the group of linear fractional transformations in a Galois field," communicated November 2, 1902 to the London Mathematical Society, the writer shows that the group G is generated by three operators subject to the relations

$$(9) \quad T^2 = I, \quad S_1^3 = I, \quad S_j^3 = I, \quad S_1S_j = S_jS_1,$$

$$(10) \quad (S_1T)^3 = I, \quad (S_jT)^4 = I, \quad (S_jS_1TS_j^{-1}S_1T)^2 = I.$$

From these we obtain relations (1), (2), (3), if we set

$$a = T, \quad b = S_jT, \quad c = S_1.$$

This is evident for relations (1). Also,

$$\begin{aligned} (ab^{-1}ab)^3 &= (S_j^{-1}TS_jT)^3 = S_j^{-1} \cdot TS_jTS_j \cdot S_jTS_jT \cdot S_j^{-1}TS_jT \\ &= S_j^{-1} \cdot S_j^{-1}TS_j^{-1}T \cdot TS_j^{-1}TS_j^{-1}T \cdot S_j^{-1}TS_jT = (S_jT)^4 = I. \end{aligned}$$

Also (2₂) and (3₂) follow from (9) and $(S_jT)^4 = I$, while (3₁) follows from (9) and the first and third relations (10). We thus obtain a new proof that (9) and (10) define G .

We may readily derive directly from (9) and (10) a complete set of relations between the two generators $b = S_jT$ and $c = S_1$ of G . We note that, from (10₁),

$$T = S_1TS_1TS_1 = S_1 \cdot TS_j^{-1} \cdot S_1 \cdot S_jTS_1 = cb^{-1}cbc.$$

We therefore have

$$S_1 = c, \quad T = cb^{-1}cbc, \quad S_j = bc^{-1}b^{-1}c^{-1}bc^{-1}.$$

Then $(S_jT)^4 = I$ follows from (5₁), $S_1^3 = I$ from (5₂), $T^2 = I$ from (5₃), $S_1S_j = S_jS_1$ from (5₃). Thus

$$\begin{aligned} S_1S_j &= cbc^{-1}b^{-1} \cdot c^{-1}bc^{-1} = bcb^{-1}c^{-1} \cdot c^{-1}bc^{-1} \\ &= bc \cdot b^{-1}cbc^{-1} = bc \cdot cb^{-1}c^{-1}b = S_jS_1. \end{aligned}$$

Since $S_1 T = c^{-1} b^{-1} c b c$ is the transform of c by $b c$, it is of period three.

The final relation (10) becomes

$$\begin{aligned}(b c^{-1} b^{-1} c \cdot b^{-1} c b c)^2 &= (c^{-1} b c b^{-1} \cdot b^{-1} c b c)^2 = (c^{-1} b c b^2 c b c)^2 \\ &= c^{-1} b (c b^2)^4 b^{-1} c = I.\end{aligned}$$

Since S_j is commutative with S_1 , the condition $S_j^3 = I$ follows from $(b^{-1} c^{-1} b^2 c^{-1})^3 = I$ or $(c b^2 c b)^3 = I$.

THE UNIVERSITY OF CHICAGO,
December 11, 1902.

NOTE ON A PROPERTY OF THE CONIC SECTIONS.

BY PROFESSOR H. F. BLICHFELDT.

(Read before the San Francisco Section of the American Mathematical Society, December 20, 1902.)

It is easily proved that if P, Q, R are any three points on the conic $Ax^2 + By^2 = 1$, and O the center of the conic, then the areas of the triangles OPQ, OPR, OQR will satisfy an equation independent of the position of the points P, Q, R . If a, b, c are the areas in question, this equation is

$$(1) \quad a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 16ABa^2b^2c^2 = 0.$$

Now we can prove that such an invariant relation is possible for no plane curves except the central conics; i. e., if we seek a plane curve C and a point O in its plane such that, if P, Q, R are any three points on C , the triangles OQR, ORP, OPQ are connected by a relation independent of the coördinates of the points P, Q, R , we find C to be a central conic section and O its center.

To prove this theorem, let O be the origin of coördinates, and let the coördinates of P, Q, R be respectively $x_1, y_1; x_2, y_2; x_3, y_3$. Then twice the areas of the three triangles are

$$\begin{aligned}2a &= \pm (y_2x_3 - y_3x_2), & 2b &= \pm (y_3x_1 - y_1x_3), \\ 2c &= \pm (y_1x_2 - y_2x_1),\end{aligned}$$

which expressions are functions of the three independent variables x_1, x_2, x_3 ; y being considered a given function of x for points on the curve.

As a, b, c must satisfy a relation independent of x_1, x_2, x_3 , the Jacobian $\partial(a, b, c)/\partial(x_1, x_2, x_3)$ must vanish. If y'_1 represents dy_1/dx_1 , etc., we find

$$y'_s\{y_s[x_1y_2 - x_2y_1 + x_1x_2(y'_2 - y'_1)] + x_3(x_2y_1y'_1 - x_1y_2y'_2)\} \\ + x_3[(x_1y_2 - x_2y_1)y'_1y'_2 + y_1y_2(y'_2 - y'_1)] + y_s(x_2y_1y'_1 - x_1y_2y'_2) = 0,$$

say

$$(2) \quad y'_s(y_sk + x_sl) + x_sm + y_sl = 0,$$

k, l, m being functions of x_1, x_2 only, and therefore independent of x_3 .

Two cases (α) and (β) may now present themselves as follows:

(α) The functions k, l, m are not all identically zero. In this case the equation (2) gives, when integrated,

$$(3) \quad y_s^2k + 2y_sx_sl + x_s^2m = f(x_1, x_2).$$

If we give to x_1 and x_2 arbitrary constant values, the equation (3) represents a conic section with its center at O .

(β) The functions k, l, m , are all zero. We must then have $x_2y'_1 - y_2 = 0$. Giving to y'_1 a definite constant value, we obtain the equation of a straight line—a special case of (3).

The theorem stated above is therefore proved.

It may be noticed that $f(x_1, x_2)$ in (3) may be multiple valued. The equation will then represent a series of similar conics similarly placed. If these are finite in number, say n , we find that, if P, Q, R be located anywhere on this system of curves, the areas a, b, c of the three triangles considered will satisfy an equation of degree

$$6n + 18n(n-1) + 6n(n-1)(n-2)$$

at most, whose left-hand member is composed of factors of form similar to (1), as the reader may prove without much difficulty.

STANFORD UNIVERSITY,
November, 1902.

THE ANALYTIC THEORY OF DISPLACEMENTS.

BY R. W. H. T. HUDSON, M.A.

Introduction.

THE subject of displacements is connected with several branches of pure mathematics and can be approached from many sides. The most comprehensive point of view is, no doubt, to regard the subject as an illustration of the theory of groups, the group here involved being a particular case of the group of linear homogeneous transformations. Again, instead of the symbol of an infinitesimal transformation, we may use the matrix of the corresponding linear substitution, and by means of elementary properties of matrices much of the work may be shortened and simplified.

By a generalization of the ordinary notion, a displacement may be defined as a projective point transformation which does not alter distance as measured with reference to an absolute quadric locus, and which further is capable of being represented as the result of an infinite number of infinitesimal transformations of the same kind. The analytic representation of a displacement must be a linear transformation of the variables which leaves the absolute unaltered. Since by a suitable choice of real or imaginary coördinates the equation of the absolute may be expressed by means of the sum of the squares of the coördinates, the problem in its simplest algebraic form is reduced to finding the most general orthogonal transformation.

The main object held in view throughout this paper is to obtain the general parametric representation of a displacement in two or three dimensions on the basis of the definition given above, and the method adopted is to integrate the equations of infinitesimal transformation. The formulæ are developed in a natural manner from the beginning and in addition to elementary processes use is made of matrices and quaternions. Not much is said about the other kind of transformation, corresponding to an improper orthogonal matrix, that is, one whose determinant is negative, because the general transformation of this kind can be obtained by combining a suitable displacement with any particular reflexion, and accordingly the analytic

representation requires no further development; again since an improper orthogonal matrix can be converted into a proper one by adding another row and column, so a transformation which preserves distance but changes order can be regarded as a displacement in a space one dimension higher.

From the limited nature of the subject it is not to be expected that any really new theorem remains to be discovered, but the method of obtaining the equations of finite displacement, though a perfectly obvious one to those who are accustomed to work with matrices, does not seem hitherto to have been successfully applied to this particular purpose.* It is hoped that the formulæ will enable the reader to obtain a clearer grasp and easy proofs of the many interesting geometric theorems given in the text-books;† and also to illustrate that aspect of the subject which is expounded in treatises on continuous groups.‡

For convenience of manipulation the subject takes the form of the theory of screws in elliptic space,§ and may form an introduction to non-euclidean kinematics. This, however, is not essential and the analysis is intended to cover all systems of measurement with the help of known methods of extension and transition.

I. Displacements in Two Dimensions.

1. In space of two dimensions the absolute is taken to be

$$x_1^2 + x_2^2 + x_3^2 = 0;$$

and since the coördinates are homogeneous we may suppose that for points not on this locus

$$x_1^2 + x_2^2 + x_3^2 = 1.$$

* Cf. Beez, *Schlömilch's Zeitschrift*, vol. 43, pp. 65, 121, 227. Stringham, *Compte Rendu du 2^me Congrès International de Mathématiciens tenu à Paris*, 1900, p. 327. Cole, "Rotations in space of four dimensions," *Amer. Journ.*, vol. 12, p. 191.

† *E. g.*, Schönflies, "Geometrie der Bewegung"; see also Study, *Math. Annalen*, vol. 39, p. 441.

‡ *E. g.*, Lie, *Transformationsgruppen*, III, p. 198; see also *Proc. L. M. S.* Burnside, vol. 26, p. 33, and Baker, vol. 34, pp. 91 and 347; these two papers appeared after the above was written.

§ See Buchheim, *Proc. L. M. S.*, vol. 15, p. 83, where Grassmann's methods are used, and vol. 16, p. 15; vol. 17, p. 240.

The general linear transformation is represented in matrix notation by

$$x' = Ax = xA',$$

where A is a three-rowed matrix and A' is its conjugate. The condition

$$x_1'^2 + x_2'^2 + x_3'^2 = x_1^2 + x_2^2 + x_3^2$$

gives

$$xA'A x = xEx,$$

where E is the unit matrix, and since this is to hold for all values of x ,

$$A'A = E;$$

and accordingly A is an *orthogonal* matrix.* Since $|A|^2 = 1$, distinct cases arise when $|A|$ is $+1$ or -1 . Since the matrix of an infinitesimal transformation differs only slightly from E it must belong to the former class, i. e., be *proper*. We shall find that the most general matrix of this class can be obtained by integrating the equations of infinitesimal transformation, and the finite transformation so obtained will be called a *displacement*.

Putting $A = E + P\delta t$ and neglecting δt^2 the condition for an orthogonal matrix is

$$(E + P'\delta t)(E + P\delta t) = E,$$

giving

$$P' + P = 0,$$

so that P is a *skew* matrix. The corresponding infinitesimal transformation is

$$x' = x + Px\delta t,$$

which may be written in the form of a differential equation

$$\frac{dx}{dt} = Px.$$

Integrating this on the hypothesis that P is independent of t ,

* For a somewhat similar development of the subject see Study, *Mittheilungen aus dem naturwissensch. Verein für Neu-Vorpommern und Rügen in Greifswald*, 1899, p. 1.

and taking x for initial and x' for final values, we find that the finite transformation is*

$$x' = e^{tP}x.$$

To obtain the actual formulæ we must expand e^{tP} in powers of P and reduce to polynomial form by means of the equation satisfied by P .

Let

$$P = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

and

$$s^2 = a^2 + b^2 + c^2.$$

Then the matrix P satisfies the cubic equation

$$\begin{vmatrix} -P & -c & b \\ c & -P & -a \\ -b & a & -P \end{vmatrix} = 0$$

or

$$P^3 + s^2P = 0,$$

whence

$$\begin{aligned} e^{tP} &= E + tP + \frac{1}{2} t^2 P^2 - \frac{1}{3!} t^3 s^2 P - \frac{1}{4!} t^4 s^2 P^2 + \dots \\ &= E + s^{-1} \sin st P + s^{-2} (1 - \cos st) P^2. \end{aligned}$$

Since this matrix contains three independent parameters $a : b : c$ and t it must be the most general matrix of the class considered, for the nine elements of an orthogonal matrix are subjected to six relations.

It is evident that the point given by $Px = 0$ is unaltered by the transformation, which may therefore be called a *rotation* about the point $a : b : c$. Let r be the *distance* between any point x and the centre of rotation $a : b : c$; then by the ordinary formula of elliptic geometry

$$\cos r = s^{-1} (ax_1 + bx_2 + cx_3),$$

* By repeated differentiation $d^n x/dt^n = P^n x$, and the integral is simply Taylor's expansion. The expression of every real proper orthogonal substitution in this form is due to Taber, *Bull. of the N. Y. Math. Soc.*, vol. 3, p. 251.

and the distance between x and $x + \delta t P x$ is

$$\delta t \{ (bx_3 - cx_2)^2 + (cx_1 - ax_3)^2 + (ax_2 - bx_1)^2 \}^{\frac{1}{2}} = \delta t \cdot s \sin \theta.$$

Hence the *angle* of rotation is $\delta t \cdot \sin \theta$, say. We may now for convenience put $s = 1$ and then, since

$$E + P^2 = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix},$$

the first equation included in $x' = e^{\theta P} x$ is

$$x'_1 = x_1 \cos \theta + \sin \theta (-cx_2 + bx_3) \\ + (1 - \cos \theta) a(ax_1 + bx_2 + cx_3),$$

agreeing with the known formula for the rotation of a point in three-dimensional euclidean space about the direction $a : b : c$.*

2. We have thus obtained the general displacement in terms of three parameters; it remains to be shown how Euler's parameters arise in connection with Cayley's expression for an orthogonal matrix.

The matrix of the finite transformation is

$$e^{\theta P} = E + \sin \theta P + (1 - \cos \theta) P^2 \\ = E + 2 \cos^2 \frac{\theta}{2} \left(\tan \frac{\theta}{2} P + \tan^2 \frac{\theta}{2} P^2 \right).$$

Put $\alpha = a \tan \frac{\theta}{2}, \quad \beta = b \tan \frac{\theta}{2}, \quad \gamma = c \tan \frac{\theta}{2},$

$$\Pi = \tan \frac{\theta}{2} P = \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}.$$

* For other methods of obtaining this formula see *Messenger of Math.*, Bromwich, vol. 29, p. 41, and Hudson, vol. 31, p. 151.

Then

$$\Pi^3 + \tan^2 \frac{\theta}{2} \Pi = 0,$$

$$\tan^2 \frac{\theta}{2} (E - \Pi) + E - \Pi^3 = \sec^2 \frac{\theta}{2} E,$$

$$(E - \Pi) \left(\sec^2 \frac{\theta}{2} E + \Pi + \Pi^2 \right) = \sec^2 \frac{\theta}{2} E.$$

$$\begin{aligned} \therefore e^{\theta P} &= -E + 2(E - \Pi)^{-1} \\ &= (E + \Pi)(E - \Pi)^{-1}, \end{aligned}$$

which is Cayley's form; the elements of Π are the parameters of the displacement. This expression is not valid when $\theta = \pi$; the matrix of the displacement then is $E + 2P^2$, and represents a half-turn about, or a reflexion in, the point a, b, c .

Conversely, when the orthogonal matrix A of the displacement is given, and $|A + E| \neq 0$, the parameters are the elements of the skew matrix $(A + E)^{-1}(A - E)$ and from them the centre and angle of rotation can be found.

3. The preceding analysis applies equally well to the displacement of a point in a plane in which the system of measurement is elliptic and to the displacement of a point on a sphere in ordinary space, and thence by a limiting process, to displacements in a euclidean plane. This process consists in putting $x_3 = 1$ and neglecting squares and products of the other coördinates. Taking

$$P = \begin{bmatrix} 0 & -1 & b \\ 1 & 0 & -a \\ -b & a & 0 \end{bmatrix}, \quad E + P^2 = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 1 \end{bmatrix},$$

we easily find that the formulæ reduce to

$$x'_1 = a + \cos \theta (x_1 - a) - \sin \theta (x_2 - b),$$

$$x'_2 = b + \sin \theta (x_1 - a) + \cos \theta (x_2 - b),$$

$$x'_3 = 1 = x_3,$$

which are the ordinary formulæ for rotation about the point a, b .

Since every improper orthogonal three-rowed matrix, for which the determinant is -1 , is the product of a proper mat-

rix and the matrix $-E$, we have to consider the meaning of the transformation $x'_1 = -x_1$, $x'_2 = -x_2$, $x'_3 = -x_3$. In the elliptic plane the points x and $-x$ are not distinct, or only nominally so, and all the transformations come under the head of displacements.* On the euclidean sphere the points x and $-x$ are diametrically opposite and the transformation $x' = -x$ is a *reflexion* in the centre. This combined with any rotation gives the most general transformation of the second kind.

In the euclidean plane the transformation $x' = -x$ is excluded by the hypothesis $x_3 = 1$, but every improper orthogonal matrix can be derived from a proper matrix by multiplying it by the improper matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which corresponds to a *reflexion* in the line $x_1 = 0$.

It is beyond the scope of this paper to develop the many interesting geometric theorems which can be proved very easily from these formulæ; we are concerned rather with the various methods of parametric representation.

4. The use of quaternions as a convenient method of formulating two-dimensional displacements may now be explained, for it follows easily from Cayley's expression for an orthogonal matrix. Since

$$(1 + i\alpha + j\beta + k\gamma)(iz_1 + jz_2 + kz_3) = -(az_1 + \beta z_2 + \gamma z_3) \\ + i(z_1 - \gamma z_2 + \beta z_3) + j(z_2 - az_3 + \gamma z_1) + k(z_3 - \beta z_1 + az_2),$$

the components of the vector part are the quantities $(E + \Pi)z$; thus, if we write

$$q = 1 + i\alpha + j\beta + k\gamma, \quad \zeta = iz_1 + jz_2 + kz_3,$$

$Vq\zeta$ is equivalent to $(E + \Pi)z$, and similarly $V\zeta q$ is equivalent to $(E - \Pi)z$. Now the general orthogonal transformation

$$x' = (E + \Pi)(E - \Pi)^{-1}x$$

* For a discussion of this peculiarity of the elliptic plane see Klein, *Nicht-Euklidische Geometrie*, I, p. 103.

can be divided into two parts

$$x' = (E + \Pi)z, \quad x = (E - \Pi)z,$$

which, when expressed in quaternion notation, are

$$\xi' = Vq\xi = q\xi - Sq\xi, \quad \xi = V\xi q = \xi q - S\xi q,$$

from which, on eliminating ξ ,

$$\xi' = q\xi q^{-1}.$$

We notice that q may be multiplied by any scalar without affecting the formula; one of the most convenient forms to take is

$$\alpha = \cos \frac{\theta}{2} q = \cos \frac{\theta}{2} + (ia + jb + kc) \sin \frac{\theta}{2},$$

$$= \alpha_0 + i\alpha_1 + j\alpha_2 + k\alpha_3,$$

say, so that $T\alpha = 1$ and then

$$\alpha^{-1} = \cos \frac{\theta}{2} - (ia + jb + kc) \sin \frac{\theta}{2}$$

$$= \alpha_0 - i\alpha_1 - j\alpha_2 - k\alpha_3.$$

The composition of rotations is effected very simply when the quaternion notation is used; for the transformation $x'' = \beta x' \beta^{-1}$ following $x' = \alpha x \alpha^{-1}$ leads to $x'' = \beta \alpha x \alpha^{-1} \beta^{-1} = \gamma x \gamma^{-1}$, where $\gamma = \beta \alpha$.

The geometric interpretation of this is the well-known theorem that successive rotations $2\pi - 2A$, $2\pi - 2B$ about the corners A , B of a spherical triangle are equivalent to a rotation $2C - 2\pi$ about the corner C .

5. One other important representation of rotations, namely as a linear transformation of the parameters of the generators of a unit sphere, is easily deduced from the quaternion formula

$$\xi' = q\xi q^{-1}$$

as follows.* We have at once $\xi' q = q\xi$. Now, on replacing k by ji or $-ji$,

* See Klein, *Lectures on the Ikosahedron*, Chap. II, § 2, and Cayley, *Math. Annalen*, vol. 15 (1879), p. 238. That a quaternion is equivalent to a two-rowed matrix was recognized by Sylvester, *Phil. Mag.*, vol. 16 (1883), p. 394.

$$\xi = ix_1 + (x_2 + ix_3)j, \quad q = 1 + i\alpha + j(\beta - i\gamma).$$

After multiplying out and replacing j^2 by -1 , we may equate coefficients of j and obtain

$$\begin{aligned} x'_1(i - \alpha) - (x'_2 + ix'_3)(\beta - i\gamma) &= (i - \alpha)x_1 - (\beta + i\gamma)(x_2 - ix_3) \\ &\quad - ix'_1(\beta - i\gamma) + (x'_2 - ix'_3)(1 + i\alpha) \\ &= (\beta - i\gamma)ix_1 + (1 - i\alpha)(x_2 - ix_3), \end{aligned}$$

where now i may be regarded as equivalent to $\sqrt{-1}$; these may be altered to

$$\begin{aligned} (1 + x'_1)(i - \alpha) - (x'_2 + ix'_3)(\beta - i\gamma) &= (i - \alpha)(1 + x_1) - (\beta + i\gamma)(x_2 - ix_3), \\ (1 - x'_1)(\beta - i\gamma) - (x'_2 - ix'_3)(i - \alpha) &= (\beta - i\gamma)(1 + x_1) - (i + \alpha)(x_2 - ix_3), \end{aligned}$$

whence by division

$$-\frac{x'_2 - ix'_3}{1 + x'_1} = \frac{(i + \alpha)(x_2 - ix_3) + (\beta - i\gamma)(1 + x_1)}{(\beta + i\gamma)(x_2 - ix_3) - (i - \alpha)(1 + x_1)},$$

which is the linear transformation of $(x_2 - ix_3)/(1 + x_1)$ referred to.

II. Infinitesimal Displacements in Three Dimensions.

6. Let A be the symmetric matrix associated with the absolute, so that its equation is

$$xAx = 0.$$

For convenience we shall suppose that $|A| = 1$ and that the actual values of the coördinates of any point not on the absolute are so chosen that $xAx = 1$.

The transformation $x' = Bx$ leaves the absolute unchanged if, for all values of x ,

$$x'Ax' = xAx;$$

this requires

$$B'AB = A,$$

where B' is the matrix conjugate to B . For a small displacement, B differs slightly from the unit matrix, so let $B = E + \epsilon C$, where ϵ^2 may be neglected; then

$$C'A + AC = 0,$$

showing that AC is a skew matrix P , say, and then $C = A^{-1}P$ and the general infinitesimal displacement is

$$x' = x + \epsilon A^{-1}Px$$

where P is any skew matrix.

In this displacement the point x moves a small distance in the direction of the point $A^{-1}Px$. Now with any skew matrix P may be associated a certain screw, or linear complex, such that Px is the null plane of x . Hence $A^{-1}Px$ is the absolute pole of the null plane of x .

The polar screw Q is obtained by reciprocating with respect to the absolute. Thus $A^{-1}Px$ is the null point of the plane Ax with reference to Q , whence

$$A^{-1}P = Q^{-1}A,$$

so that the motion of x may be described as being towards the null point with reference to the polar screw of its absolute polar plane

The motion of any plane l is given by

$$l'x' = lx, \quad x' = (E + \epsilon A^{-1}P)x;$$

whence

$$l' = l(E + \epsilon A^{-1}P)^{-1} = l(E - \epsilon A^{-1}P) = l + \epsilon PA^{-1}l,$$

on forming the conjugate matrix, so that the plane l turns about its intersection with $PA^{-1}l$, which is the null plane of the pole of l , and is the same as $AQ^{-1}l$ which is the polar of the null point of l with reference to the polar screw.*

7. The transformation

$$y = A^{-1}Qx, \quad y' = A^{-1}Qx'$$

transforms the line xx' into the polar line of the intersection of the null planes $u = Qx$, $u' = Qx'$. If xx' is a ray of the complex Q , i. e., if $x'Qx = 0$ then uu' is the same line; so this transformation has the property of transforming every line of a certain complex into its polar line. Consequently if a ruled surface is generated by lines belonging to the screw Q the linear point transformation whose matrix is $A^{-1}Q$ transforms the surface into its reciprocal with respect to the quadric A . †

* These theorems are given by Buchheim, *Proc. L. M. S.*, vol. 16, p. 15.

† This is a more general statement of a theorem contained in a note in the *Messenger of Math.*, vol. 29, p. 191.

If we are reciprocating with respect to the surface

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,$$

we have $A = E$ and the transformation is

$$y = Qx.$$

8. We shall consider next in some detail the case when $|P| = 0$ and the screw associated with the motion degenerates into a line. We cannot immediately apply the formulas for polar screws because the inverse matrices do not now exist. It is convenient to use two matrices in connection with a line. Let *

$$P = \begin{bmatrix} 0 & -r & +q & p' \\ +r & 0 & -p & q' \\ -q & +p & 0 & r' \\ -p' & -q' & -r' & 0 \end{bmatrix} \text{ and } P' = \begin{bmatrix} 0 & -r' & +q' & p \\ +r' & 0 & -p' & q \\ -q' & +p' & 0 & r \\ -p & -q & -r & 0 \end{bmatrix},$$

where p, q, r, p', q', r' are the coördinates of the line; then

$$PP' \equiv -(pp' + qq' + rr') E = 0.$$

Hence if Q, Q' are the matrices associated with another line, the two lines intersect if

$$PQ' + QP' = 0.$$

The lines are polar if

$$Q = AP'A, \quad Q' = A^{-1}PA^{-1} \quad (\text{since } |A| = 1),$$

for then the pole of any plane through one line lies on the other.

Hence the line P touches the absolute if it cuts Q , i. e., if

$$PA^{-1}PA^{-1} + AP'AP' = 0.$$

For any other line we shall take this expression to be equal to -1 .

* This must not be confused with Frobenius's notation for conjugate matrices; in the case of a skew matrix P the conjugate is simply P .

To find the distance from a point x to a line P we notice that the plane Px cuts the polar line Q in the point $Q'Px$ and the distance between this and x is

$$\cos^{-1} \frac{x A Q' P x}{\sqrt{x P Q' A Q' P x}}.$$

Now

$$\begin{aligned} x P A^{-1} (P A^{-1} P A^{-1}) P x &= -x P A^{-1} P x - x P A^{-1} (A P' A P') P x \\ &= -x P A^{-1} P x; \end{aligned}$$

hence the distance from x to P is

$$\rho = \sin^{-1} \sqrt{-x P A^{-1} P x}.$$

The distance between x and its displaced position given by the transformation

$$x' = x + \epsilon A^{-1} P x$$

is

$$\sqrt{(\epsilon A^{-1} P x) A (\epsilon A^{-1} P x)} = \epsilon \sqrt{-x P A^{-1} P x} = \epsilon \sin \rho;$$

hence ϵ is the small angle of rotation about P . As in the case of two dimensions, so also here the general finite displacement is obtained by integrating

$$\frac{dx}{d\theta} = A^{-1} P x.$$

9. Returning to the general screw motion, let P be the screw and L any line; then it is possible to choose k so that $P - kL$ is a line, $= k'M$ say. Thus,

$$\epsilon A^{-1} P = \epsilon k A^{-1} L + \epsilon k' A^{-1} M$$

and the motion consists of two small rotations ϵk about L and $\epsilon k'$ about M . The most important case is when L and M are polar lines; they are then the *axes* of the screw and are of the form $P + \lambda Q$ where $|P + \lambda Q| = 0$. Discussion of this and other methods of resolution is simplified by taking the absolute in the form used in elliptic space. This will be done in connection with finite displacements.

III. *Finite Displacements in Three Dimensions.**

10. We have seen that the general infinitesimal displacement is

$$x' = x + \epsilon A^{-1}Px,$$

where ϵ is a small constant, A is the matrix of the absolute, and P is any four-rowed skew matrix. In what follows we shall take the absolute to be

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,$$

so that $A = E$. By a suitable change of coördinates the absolute can be reduced to this form, and by admitting the possibility of imaginary coördinates we are able to include all kinds of space in one investigation.

The above transformation can be written as a differential equation

$$\frac{dx}{dt} = Px,$$

where t is a quantity on whose variation the motion of the point x depends. As in the case of two dimensions the most general proper orthogonal transformation is obtained by integrating this on the hypothesis that the elements of P are constant. Taking x for the initial and x' for the final position of the moving point, we can integrate this equation in the symbolic form

$$x' = e^{tP}x,$$

and it remains to express e^{tP} in a non-transcendental form. This may be done by a general method applicable to matrices of any order.†

For if the identical equation satisfied by a matrix P is

$$f(P) \equiv (P - \lambda_1 E)(P - \lambda_2 E) \cdots (P - \lambda_n E) = 0,$$

we have, by division, assuming the latent roots to be distinct,

$$P^m \equiv (P^{m-n} + \cdots)f(P) + \sum_{i=1}^n \frac{\lambda_i^n}{f'(\lambda_i)} \frac{f(P)}{P - \lambda_i E};$$

* Equivalent to rotations about a fixed point in four dimensions; from this point of view the subject has been treated by Cole, *Amer. Journ.*, vol. 12, p. 191, and Jahnke, *Jahresbericht der deutschen Math.-Vereinigung*, April, 1902.

† See Bromwich, *Proc. Camb. Phil. Soc.*, vol. 11, p. 75, and the references there given.

whence, using $f(P) = 0$,

$$\phi(P) = \sum_{i=1}^n \frac{\phi(\lambda_i)}{f'(\lambda_i)} \cdot \frac{f(P)}{P - \lambda_i E},$$

ϕ being any function expandible in ascending powers.* In the present instance the four-rowed matrix

$$P = \begin{bmatrix} 0 & -r & q & p' \\ r & 0 & -p & q' \\ -q & p & 0 & r' \\ -p' & -q' & -r' & 0 \end{bmatrix}$$

satisfies the equation

$$P^4 + (p^2 + q^2 + r^2 + p'^2 + q'^2 + r'^2)P^2 + (pp' + qq' + rr')^2 E = 0,$$

or say

$$(P^2 - \lambda^2 E)(P^2 - \mu^2 E) = 0.$$

Then, applying the rule given above,

$$\begin{aligned} e^P &= \frac{e^\lambda}{2\lambda(\lambda^2 - \mu^2)} (P + \lambda E)(P^2 - \mu^2 E) \\ &\quad + \frac{e^{-\lambda}}{-2\lambda(\lambda^2 - \mu^2)} (P - \lambda E)(P^2 - \mu^2 E) + \text{etc.} \\ &= \frac{P^2 - \mu^2 E}{\lambda^2 - \mu^2} \left(\cosh \lambda E + \frac{1}{\lambda} \sinh \lambda P \right) \\ &\quad + \frac{P^2 - \lambda^2 E}{\mu^2 - \lambda^2} \left(\cosh \mu E + \frac{1}{\mu} \sinh \mu P \right); \end{aligned}$$

and this is the required matrix of the finite transformation after we have put $t = 1$.

11. It is evident that when the identical equation for P reduces to a binomial form, the evaluation of e^P can be effected much more easily. This happens in two cases which will now be discussed, and the general case can be reduced to either of them.

* This formula was first given by Sylvester, *Johns Hopkins Univ. Circulars*, 3 (1882), pp. 9 and 210.

Let P' be the matrix obtained from P by interchanging p and p' , q and q' , r and r' . Then P and P' correspond to polar screws. It is easily verified by actual multiplication that

$$PP' = P'P = -(pp' + qq' + rr')E,$$

$$P^2 + P'^2 = -(p^2 + q^2 + r^2 + p'^2 + q'^2 + r'^2)E.$$

The first special case to be considered is when

$$P = P'.$$

Then P is called a *right-vector* and the identical equation takes the simple form

$$P^2 + (p^2 + q^2 + r^2)E = 0.$$

Putting

$$\theta = t\sqrt{p^2 + q^2 + r^2}$$

we have

$$e^{tP} = 1 + tP - \frac{\theta^2}{2} - \frac{t\theta^2}{3!}P + \dots = \cos \theta + \frac{\sin \theta}{\theta}tP,$$

and in the corresponding displacement every point moves through the same distance θ along a right parallel of the system.*

Similarly if $P = -P'$, P is called a *left vector*, and the matrix of the displacement has the same form as before.

Now the general screw P can be expressed as the sum of a right vector and a left vector, for we have only to write

$$2U = P + P', \quad 2V = P - P',$$

and then

$$P = U + V.$$

Further

$$UV = VU,$$

so that the matrices U , V , and therefore also the corresponding finite transformations, are commutative. Hence the general displacement e^{tP} can be resolved into the succession of vector displacements e^{tU} , e^{tV} taken in either order. Accordingly, if we put

* For a discussion of vector displacements see Whitehead, *Universal Algebra*, p. 472.

$$4\alpha^2 = (p + p')^2 + (q + q')^2 + (r + r')^2,$$

$$4\beta^2 = (p - p')^2 + (q - q')^2 + (r - r')^2,$$

$$U = \alpha U_1, \quad V = \beta V_1,$$

then

$$e^P = (\cos \alpha E + \sin \alpha U_1)(\cos \beta E + \sin \beta V_1)$$

or

$$e^P = \cos \alpha \cos \beta E + \sin \alpha \cos \beta U_1 + \cos \alpha \sin \beta V_1 \\ + \sin \alpha \sin \beta U_1 V_1,$$

and this form may easily be made to agree with the preceding by putting

$$\alpha + \beta = i\lambda, \quad \alpha - \beta = i\mu.$$

12. The second special case is when

$$pp' + qq' + rr' = 0,$$

so that P represents a *line*. The corresponding displacement is a rotation about this line, and since $PP' = 0$ we deduce at once that rotations about polar lines are independent. Writing

$$c^2 = p^2 + q^2 + r^2 + p'^2 + q'^2 + r'^2,$$

we have

$$P^2 + P'^2 + c^2 E = 0,$$

and therefore

$$P^3 + c^2 P = 0.$$

Integration in this case is precisely the same as in two dimensions and we obtain for the finite equation of rotation through an angle θ

$$x' = e^{tP}x = \{1 + c^{-1} \sin \theta P + c^{-2}(1 - \cos \theta)P^2\}x.$$

Now the general screw P can be expressed as the sum of two polar lines, its axes.* These are the lines L, L' where

$$L = \sigma P + \sigma^{-1}P', \quad L' = \sigma^{-1}P + \sigma P',$$

* Whitehead, Universal Algebra, p. 401.

and $LL' = 0$; giving

$$(\sigma^2 + \sigma^{-2})(pp' + qq' + rr') + c^2 = 0.$$

Write for abbreviation

$$\omega = pp' + qq' + rr',$$

then the sum of the squares of the six elements of L is

$$(\sigma^2 + \sigma^{-2})c^2 + 4\omega, = -\omega(\sigma^2 - \sigma^{-2})^2;$$

accordingly it is convenient to introduce a new matrix M given

by
$$L = (\sigma^2 - \sigma^{-2})\sqrt{-\omega}M.$$

Then

$$P = \frac{\sigma L - \sigma^{-1}L'}{\sigma^2 - \sigma^{-2}}$$

$$= \sigma \sqrt{-\omega}M - \sigma^{-1}\sqrt{-\omega}M'.$$

Accordingly the displacement defined by P is equivalent to rotations about the lines M, M' , the angles being $\theta = \sigma \sqrt{-\omega}$ and $\theta' = -\sigma^{-1} \sqrt{-\omega}$. The matrix e^P of the finite transformation has the form

$$E + \sin \theta M + \sin \theta' M' + (1 - \cos \theta)M^2 + (1 - \cos \theta')M'^2,$$

which may be compared with either of the preceding forms.

13. As in the case of two dimensions, when $\theta \neq \pi$, $e^{\theta M}$ can be expressed in the form

$$\left(E + \tan \frac{\theta}{2} M\right) \left(E - \tan \frac{\theta}{2} M\right)^{-1}.$$

Hence, since $MM' = 0$,

$$e^P = e^{\theta M + \theta' M'} = e^{\theta M} e^{\theta' M'}$$

$$= \left(E + \tan \frac{\theta}{2} M + \tan \frac{\theta'}{2} M'\right) \left(E - \tan \frac{\theta}{2} M - \tan \frac{\theta'}{2} M'\right)^{-1},$$

so that in Cayley's expression for an orthogonal transformation the skew matrix used is

$$\tan \frac{\theta}{2} M + \tan \frac{\theta'}{2} M'.$$

This matrix is closely connected with Burnside's theorem that the general screw displacement can be resolved into two half turns about lines cutting the axes of the screw at right angles.* If P and Q are the matrices associated with any two lines whose shortest distances are θ, θ' , and if M, M' are their common normals, it is easy to verify that†

$$PQ - QP = -M \sin \theta \cos \theta' + M' \sin \theta' \cos \theta;$$

on the other hand the succession of half turns is represented by the matrix

$$(E + 2P^2)(E + 2Q^2);$$

and it may left as an exercise in matrix notation‡ to verify that this is the same as

$$(PQ + P'Q')(QP + Q'P')^{-1},$$

and that this is

$$(E - \tan \theta M + \tan \theta' M')(E + \tan \theta M - \tan \theta' M')^{-1}$$

which, as was shown above, is the matrix of a displacement on a screw whose axes are M, M' , the angles of rotation being $-2\theta, +2\theta'$ respectively.§

14. We must now establish the connection between quaternions and the parametric representation of the general displacement.|| Taking as before the skew matrix P of the displacement to be

$$\begin{bmatrix} 0 & -r & q & p' \\ r & 0 & -p & q' \\ -q & p & 0 & r' \\ -p' & -q' & -r' & 0 \end{bmatrix},$$

* *Messenger of Math.*, vol. 23, p. 19; *Proc. L. M. S.*, vol. 26, p. 33, where a simple geometric proof is given. A very short analytic proof can be given by taking the axes of the screw to be opposite edges of a tetrahedron of reference self-conjugate with respect to the absolute.

† The formulæ for common normals are given in the *Messenger of Math.*, vol. 32, p. 31.

‡ Hudson, *Messenger of Math.*, vol. 32, p. 51.

§ This theorem, treated geometrically, has formed the subject of papers by Wiener, *Leipziger Berichte*, vol. 42 (1890), Gale, *Annals of Math.*, Oct., 1900; Wood, *Annals of Math.*, July, 1901.

|| The fact that a special four-rowed matrix is equivalent to a quaternion is due to Frobenius, *Crelle*, vol. 84, p. 59.

and putting

$$\alpha_0 = \cos \alpha, \quad \alpha_1 = \sin \alpha \frac{p + p'}{2\alpha}, \quad \alpha_2 = \sin \alpha \frac{q + q'}{2\alpha},$$

$$\alpha_3 = \sin \alpha \frac{r + r'}{2\alpha},$$

so that

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1,$$

the matrix of the right vector displacement

$$x' = \left(\cos \alpha E + \sin \alpha \frac{P + P'}{2\alpha} \right) x$$

is

$$\begin{bmatrix} \alpha_0 & -\alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_3 & \alpha_0 & -\alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 & \alpha_0 & \alpha_3 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \alpha_0 \end{bmatrix},$$

and that of the left vector displacement

$$x' = \left(\cos \beta E + \sin \beta \frac{P - P'}{2\beta} \right) x$$

is

$$\begin{bmatrix} \beta_0 & -\beta_3 & \beta_2 & -\beta_1 \\ \beta_3 & \beta_0 & -\beta_1 & -\beta_2 \\ -\beta_2 & \beta_1 & \beta_0 & -\beta_3 \\ \beta_1 & \beta_2 & \beta_3 & \beta_0 \end{bmatrix},$$

where

$$\beta_0 = \cos \beta, \quad \beta_1 = \sin \beta \frac{p - p'}{2\beta}, \quad \beta_2 = \sin \beta \frac{q - q'}{2\beta},$$

$$\beta_3 = \sin \beta \frac{r - r'}{2\beta},$$

and

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1.$$

Expressed in quaternion notation these transformations are

$$x'_4 + ix'_1 + jx'_2 + kx'_3 = (\alpha_0 + i\alpha_1 + j\alpha_2 + k\alpha_3)(x_4 + ix_1 + jx_2 + kx_3)$$

and

$$x'_4 + ix'_1 + jx'_2 + kx'_3 = (x_4 + ix_1 + jx_2 + kx_3)(\beta_0 - i\beta_1 - j\beta_2 - k\beta_3)$$

respectively, which may be written

$$x' = ax \quad \text{and} \quad x' = xb^{-1},$$

all the letters representing unit quaternions. Combining these two vector displacements, we find for the general displacement either

$$x' = ax, \quad x'' = x'b^{-1} \text{ giving } x'' = axb^{-1},$$

or

$$x' = xb^{-1}, \quad x'' = ax' \text{ also giving } x'' = axb^{-1},$$

showing once more that right and left vector displacements are commutative.*

Now put

$$v = ip + jq + kr, \quad v' = ip' + jq' + kr',$$

then the formula $x' = axb^{-1}$

is the same as

$$x' = \left(\cos \alpha + \sin \alpha \frac{v + v'}{2\alpha} \right) x \left(\cos \beta - \sin \beta \frac{v - v'}{2\beta} \right)$$

$$\text{or, } x' = \left(\frac{v + v'}{2\alpha} \right)^{\frac{2\alpha}{\pi}} (x) \left(\frac{v - v'}{2\beta} \right)^{-\frac{2\beta}{\pi}},$$

and it may be noticed that

$$2\alpha = T(v + v'), \quad 2\beta = T(v - v').$$

To pass to euclidean or hyperbolic space, replace x_4 by x_4/ω and v' by $\omega v'$, and ultimately put $\omega = 0$ or i respectively.† In

* Klein, *Nicht-Euklidische Geometrie*, II, p. 123.

† For further explanation see *Messenger of Math.*, vol. 31, p. 151, where the application of biquaternions is considered.

the former case we may at once neglect ω^2 , and then the quaternion a takes the form $q + \omega q'$, where q is a unit quaternion and q' is connected with q only by the relation

$$Sq'q^{-1} = 0,$$

since $Ta = 1$. Evidently b takes the form $q - \omega q'$ and so, if we put

$$\xi = ix_1 + jx_2 + kx_3, \quad \xi' = ix'_1 + jx'_2 + kx'_3,$$

the general displacement is given by

$$\begin{aligned} \frac{x'_4}{\omega} + \xi' &= (q + \omega q') \left(\frac{x_4}{\omega} + \xi \right) (q - \omega q')^{-1} \\ &= (q + \omega q') \left(\frac{x_4}{\omega} + \xi \right) (q^{-1} + \omega r), \end{aligned}$$

where $qr - q'q^{-1} = 0$; and this is equivalent to

$$\begin{aligned} x'_4 &= x_4, \quad \xi' = q\xi q^{-1} + (qr + q'q^{-1})x_4, \\ &= q\xi q^{-1} + 2q'q^{-1}x_4. \end{aligned}$$

In this we may put $x_4 = 1$ and so obtain the general displacement in euclidean space in terms of 8 homogeneous parameters, the constituents of q and q' , connected by the two relations *

$$Tq = 1, \quad Sq'q^{-1} = 0.$$

The ends set forth in the introduction have now been achieved.

ST. JOHN'S COLLEGE,
CAMBRIDGE, ENGLAND.

* Cf. Study, *Math. Annalen*, vol. 39, p. 536.

NOTES.

THE January number (second series, volume 4, number 2) of the *Annals of Mathematics* contains the following papers: "The logarithm as a direct function," by J. W. BRADSHAW, with an introduction by W. F. OSGOOD; "On positive quadratic forms," by PAUL SAUREL; "Multiple points on Lissajous's curves in two and three dimensions," by E. A. HOOK; "A special quadri-quadric transformation of real points in a plane," by C. C. ENGBERG.

The editorial staff of the *Annals* have recently added Dr. E. V. HUNTINGTON to their number.

AT the recent meeting of Section A of the American association for the advancement of science, held at Washington, D. C., twenty-three papers were read, of which the following related to pure mathematics: By Professor L. G. WELD: "Saint Loup's linkage."—By Dr. PAUL CARUS: "The foundations of mathematics."—By Professor G. B. HALSTED: "The teaching of geometry"; "The Bolyai centenary."—By Professor E. O. LOVETT: "Special periodic solutions of the problem of n bodies"; "The problems of three or more bodies with prescribed orbits."—By Mr. J. J. QUINN: "A development of conic sections by kinematic methods."—By Professor J. S. MILLER: "Note on a geometrical analysis" (by title).—By Dr. C. J. KEYSER: "Concerning Bolzano's contributions to assemblage theory" (by title).

AT the meeting of the London mathematical society held on January 8, the following papers were read: By Professor A. LODGE: "A method of representing imaginary points by real points in a plane."—By Dr. J. LARMOR: "On the mathematical expression of the principle of Huygens."—By Professor A. E. H. LOVE: "Wave motion with discontinuities at wave points."—By Dr. H. F. BAKER: "Functions of several variables."—By Mr. W. H. YOUNG: "On non-uniform convergence and term-by-term integration of series."—By Professor L. E. DICKSON: "Generational relations for the abstract group simply isomorphic with the linear fractional group in the Galois field $[2^n]$."—By Rev. F. H. JACKSON: "Series connected with

the enumeration of partitions (second paper).”—By Mr. J. BRILL: “On the minors of a skew-symmetrical determinant.”—By Professor W. S. BURNSIDE: “On the Jacobian of two binary quantics considered geometrically”; “On the resolution of some skew invariants of binary quantics into their factors in terms of their roots.”

At the meeting of November 13, Professor HORACE LAMB was elected president, and Professors A. E. H. LOVE and W. BURNSIDE were elected secretaries of the Society.

HARVARD UNIVERSITY: At the summer school, held in July and August, there will be offered for the first time an advanced mathematical course, the subject being: The theory of functions of a complex variable. The course is given by Professors OSGOOD and BÔCHER. Like the other courses given in the summer school, it is intended to occupy the students' entire working time during the six weeks of its continuance.

THE Macmillan Company announce the early publication of the following mathematical works: Elements of the theory of integers, by JOSEPH BOWDEN; The principles of mathematics, Volume 1, by BERTRAND RUSSELL; A treatise on determinants, by R. F. SCOTT, new edition edited by G. B. MATHEWS; The algebra of invariants, by J. H. GRACE and A. YOUNG; A treatise on spherical astronomy, by R. S. BALL. Longmans, Green and Company have in press: A first course in the infinitesimal calculus, by D. A. MURRAY.

THE Hungarian academy of sciences announces two mathematical prizes of 2000 crowns each, the first being offered for a manual of absolute geometry, of about 25 folios, to be submitted before December 31, 1904; the second for a study of a class of differential invariants, to be submitted before December 31, 1903.

THE annual register of the German Mathematiker-Vereinigung, issued in January, shows that forty-nine Americans are members of the society, thirteen having been elected during the year 1902.

PROFESSOR H. POINCARÉ has been made a commander of the Legion of honor.

PROFESSOR A. R. FORSYTH has been elected president of the Mathematical association.

THE College entrance examination board has appointed as examiners in mathematics for 1903 Professor CHARLOTTE A. SCOTT, Professor W. H. METZLER and Mr. J. S. FRENCH.

DR. O. D. KELLOGG has been appointed instructor in mathematics at Princeton University.

SIR GEORGE GABRIEL STOKES, master of Pembroke college, Cambridge, Lucasian professor of mathematics since 1849, died on February 1. He was born at Skreen, Ireland, in 1819, and was educated at Pembroke, taking his degree in 1841, being senior wrangler. He was president of the Royal society from 1885 to 1890, and from 1887 to 1892 he represented Cambridge University in parliament.

JAMES GLAISHER, the meteorologist, aéronaut and mathematician, died at London on February 8. He was born at London April 7, 1809. In 1829 he was employed in the ordnance survey of Ireland, in 1833 he became assistant at the Cambridge observatory, in 1836 he was appointed assistant in the Greenwich observatory, and in 1840 he became superintendent of the magnetical and meteorological department. He was a fellow of the Royal society (1849) and president of the Meteorological society, and was connected with various other learned associations. In mathematics he is chiefly known for his work on factor tables.

REV. H. W. WATSON, D.Sc., F.R.S., at one time mathematical lecturer at King's College, London, and Trinity College, Cambridge, died on January 11, aged seventy-five years.

LAWRENCE SLUTER BENSON, for many years an ardent expounder of perverse mathematical doctrine, died at an advanced age at Newark, N. J., January 27.

A SIXTH edition of his catalogue of mathematical models for use in advanced mathematical instruction, has just been issued by Martin Schilling, Halle a. S.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- BOHREN (A.). Ueber die Fresnelschen Integrale. Bern, 1901. 8vo. 48 pp.
- ECHOLS (W. H.). An elementary text-book on the differential and integral calculus. New York, Holt, 1902. 8vo. 10 + 480 pp. Cloth. \$2.00
- FREGE (G.). Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet. Vol. II. Jena, Pohle, 1903. 8vo. 15 + 266 pp. M. 12.00
- JECKLIN (L.). Historisch-kritische Untersuchung über die Theorie der hypergeometrischen Reihe bis zu den Entdeckungen von E. E. Kummer. Bern, 1901. 8vo. 87 pp.
- JUNKER (F.). Höhere Analysis. Teil I: Differentialrechnung. 2te, verbesserte Auflage. 2ter Abdruck. Leipzig, Göschen, 1902. 12mo. 231 pp. Cloth. (Sammlung Göschen, No. 87.) M. 0.80
- KREBS (A.). Konstruktionen gleichschenkliger Dreiecke mit Hilfe von Kurven höherer Ordnung. Bern, 1902. 8vo. 95 pp.
- KUMMER (M.). Darlegung der Weberschen und verwandter Integrale, ihre Theorie und Anwendung. Bern, 1902. 8vo. 63 pp.
- MACFARLANE (A.). A report on recent progress in the quaternion analysis. (*Proceedings of the American Association for the Advancement of Science*, Vol. 51, pp. 305-326. 1902.) 8vo.
- NEWSON (H. B.). Report on the theory of collineations. (*Proceedings of the American Association for the Advancement of Science*, Vol. 51, pp. 579-599. 1902.) 8vo.
- PALÁGYI (M.). Kant und Bolzano. Eine kritische Parallele. Halle, 1902. 8vo. 11 + 124 pp. M. 3.00
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- SALMON (G.). Traité de géométrie analytique (courbes planes), destiné à faire suite au traité des sections coniques. Ouvrage traduit de l'anglais par O. Chemin, et suivi d'une étude sur les points singuliers par G. Halphen. 2e tirage. Paris, Gauthier-Villars, 1903. 8vo. 19 + 667 pp. Fr. 12.00
- SCHOUTEN (G.). Inleiding tot de studie der elliptische functionen van Weierstrass. Delft, 1902. 8vo. 152 pp. 3 fl. 90c.
- SITZUNGSBERICHTE der Berliner Mathematischen Gesellschaft. Herausgegeben vom Vorstand der Gesellschaft. 1ster Jahrgang. Leipzig, Teubner, 1902. 8vo. 4 + 66 pp. M. 2.40
- SPIESS (O.). Die Grundbegriffe der Iterations-Rechnung. Basel, 1902. 8vo. 34 pp.

STAEBLE (F.). Untersuchung der Flächen, deren Krümmungslinien bei orthogonaler Projektion auf eine andere Fläche wieder Krümmungslinien werden. (Diss.) München, 1901. 8vo. 31 pp.

STOUFF (X.). Sur la première lettre arithmétique d'Hermite à Jacobi. (*Bulletin des sciences mathématiques*, (2) 26, 1902.) 8vo. 7 pp.

STUDY (E.). Geometrie der Dynamen. Die Zusammensetzung von Kräften und verwandte Gegenstände der Geometrie. Leipzig, Teubner, 1903. 8vo. 13 + 603 pp. M. 21.00

THUE (A.). Om en pseudomekanisk metode i geometrien. Kristiania, 1902. 8vo. 111 pp. 1 kr. 75 ø.

II. ELEMENTARY MATHEMATICS.

ADAM (V.). Taschenbuch der Logarithmen für Mittelschulen und höhere Bildungsanstalten. 31ste Auflage. Stereotypausgabe. Wien, Mejsstrik, 1903. 4to. 10 + 100 pp. Cloth. M. 1.00

ASHTON (C. H.) and MARSH (W. R.). Five place logarithmic tables together with a four place table of natural functions. New York, Scribner, 1902. 12mo. 21 + 93 pp. Cloth. \$0.60

BARBISCH (H.). See JAHNE (J.).

BLAINE (R. G.). Some quick and easy methods of calculating: a simple explanation of the theory and use of the slide-rule, logarithms, etc. 2d edition, revised and enlarged. London, Spon, 1902. 12mo. 164 pp. Cloth. 2s. 6d.

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COMAS Y MUNTANER (J.). Programa de nociones y ejercicios de aritmética y geometría. Santiago, Tipografía Galaica, 1902. 8vo. 16 pp. (Instituto general y técnico de Santiago.) Fr. 0.50

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- KLEINSCHMIDT (E.). Leitfaden der Geometrie und des geometrischen Zeichnens für Mädchen-Bürgerschulen. Teil I: Ite Klasse. 3te Auflage. Inhaltlich unveränderter, nach der neuen Rechtschreibung hergestellter Abdruck der 2ten Auflage. Wien, Hölzler, 1902. 8vo. 3 + 67 pp. Cloth. M. 1.00
- KÖLTZSCH (A.). Grundzüge der Raumlehre. Ein Lern- und Uebungsbuch zum Gebrauche in Volksschulen, Fortbildungsschulen, Präparanden-Anstalten und Mittelschulen. Heft I. Mit etwa 500 Uebungsaufgaben. 3te, verbesserte Auflage. Leipzig, Merseburger, 1903. 8vo. 78 pp. M. 0.60
- LONDON University intermediate mathematics; guide to intermediate examinations in arts and science. 6th edition. London, Clive, 1902. 8vo. 120 pp. 2s.
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OF MATHEMATICAL SCIENCE

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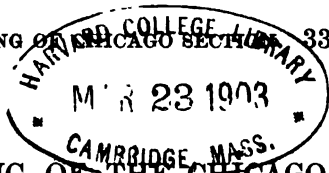
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[April, 1903.] JANUARY MEETING OF THE CHICAGO SECTION 337



THE JANUARY MEETING OF THE CHICAGO SECTION.

THE twelfth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Friday and Saturday, January 2-3, 1903, the first session opening at 10:30 A. M. About thirty persons were in attendance, including the following members of the Society:

Professor Oskar Bolza, Professor D. F. Campbell, Professor E. W. Davis, Professor J. F. Downey, Professor Thomas F. Holgate, Dr. H. G. Keppel, Professor Kurt Laves, Professor H. Maschke, Professor E. H. Moore, Dr. F. R. Moulton, Professor H. B. Newson, Miss Ida M. Schottenfels, Professor J. B. Shaw, Dr. S. E. Slocum, Professor Henry S. White.

In the absence of the President or a Vice-President of the Society, Professor H. B. Newson was elected chairman of the Section. At the first session the Secretary reported that since the organization of the Section in 1897, one hundred and sixty-three papers had been read before it, by sixty-three different persons; and of these, seventy-two papers had been published. Later, during the meeting, the retiring President of the Society, Professor Moore, favored the Section with the presentation of his presidential address which had been delivered at the annual meeting in New York on December 30.

The following officers of the Section were elected for the ensuing year:

Secretary, Professor THOMAS F. HOLGATE.

Additional members of the programme committee, Professor E. B. SKINNER, Dr. S. E. SLOCUM.

The report of the committee appointed at the last Christmas meeting to devise a scheme of equivalent requirements for candidates proceeding to their second academic degree, with mathematics as their major subject, was taken up and discussed, final action being postponed until the April meeting. The report deals with the undergraduate programme which should be accepted as a basis for graduate work, and outlines in general terms the character of the work which should be demanded during the first year of graduate study. Copies of the report

are in the hands of the Secretary for distribution to members of the Society.

The following papers were read :

(1) Dr. SAUL EPSTEEN : "Determination of the group of rationality of a differential equation."

(2) Professor E. W. DAVIS : "A group in logic."

(3) Professor H. B. NEWSON : "On the generation of finite from infinitesimal transformations ; a correction."

(4) Professor L. E. DICKSON : "The ternary orthogonal group in a general field."

(5) Professor L. E. DICKSON : "The group defined for a general field by the rotation groups."

(6) Professor A. S. HATHAWAY : "Vector analysis."

(7) Professor J. B. SHAW : "On nilpotent algebras."

(8) Professor D. F. CAMPBELL : "On homogeneous quadratic relations in the solution of a linear differential equation of the fourth order."

(9) Dr. S. E. SLOCUM : "Relation between real and complex groups with respect to their structure and continuity."

(10) Professor ARNOLD EMCH : "On the involution of stresses in a plane."

(11) Mr. R. E. WILSON : "Polar triangles of a conic and certain circumscribed quartic curves" (preliminary communication).

(12) Professor H. S. WHITE : "Orthogonal linear transformations and certain invariant systems of cones" (preliminary communication).

(13) Professor R. E. ALLARDICE : "On the envelope of the axes of similar conics through three fixed points."

Mr. Wilson was introduced by Professor White and Dr. Epsteen by Professor Maschke. In the absence of the authors, Professor Emch's paper and Professor Dickson's two papers were read by title; Professor Allardice's paper was read by Professor White, and Professor Hathaway's by Professor Shaw.

Professor Davis's paper appears in the present number of the BULLETIN. Professor Allardice's paper was published in the January number of the *Transactions*. Abstracts of the other papers are as follows :

1. Dr. Epsteen pointed out that when a special algebraic equation is given, its group can be theoretically determined 1° by constructing a $n!$ valued function and finding the Galois

resolvent, or 2° by applying the characteristic double property of the group, viz.: every rational function of the roots which remains unaltered by all the substitutions of G lies in R ; and conversely. Both of these methods are generally impracticable, and in practice some device is resorted to in order to obtain the group. When a single rational function is known the following theorem has been found very useful: "If a rational function $\psi(x_1, \dots, x_n)$ remains formally unaltered by the substitutions of a group G' and by no other substitutions, and if ψ equals a quantity lying in R and if the conjugates of ψ under G_n are all distinct, then the group of the given equation for the domain R is a subgroup of G' (cf. Dickson, *Theory of algebraic equations*, which is soon to appear, page 56).

The situation is exactly the same when we deal with linear differential equations. When a special linear differential equation is given, its group may be found: 1° by constructing the Picard resolvent, or 2° by applying the Picard-Vessiot characteristic double property of the group, viz.: every rational differential function (of a fundamental system) of the integrals which remains unaltered as a function of x , by the most general transformation of G lies in R and conversely. In this case these methods are even more impracticable than for the algebraic equations, and in order to determine the group a device should be resorted to. With this end in view the author has proved the following theorem: "If a rational differential function (of a fundamental system) of the integrals

$$\psi(y_1, \dots, y_n; y'_1, \dots, y'_n; \dots; y_1^{(n-1)}, \dots, y_n^{(n-1)})$$

remains formally unaltered by the transformations of a complex r -parameter linear homogeneous group G_r and if, when ψ is transformed by the most general linear homogeneous transformation, the resulting function contains $n^2 - r$ essential parameters, then the group of the given equation G is a subgroup of G_r ."

4. The first paper by Professor Dickson is a study of the structure of the ternary orthogonal group in an arbitrary realm of rationality. The problem had been solved in two cases: for continuous fields by Weber (*Algebra*, II, second edition, pages 244-54); for finite fields by the writer (*Linear Groups*, page 164). The group of all ternary orthogonal transformations of

determinant unity in a field F is designated $O(3, F)$. If $i = \sqrt{-1}$ belongs to F , $O(3, F)$ is simply isomorphic with the group of all linear fractional transformations in F . The latter has as an invariant subgroup the simple group of all linear fractional transformations in F of determinant unity. Simple generators of $O(3, F)$ are given. If i extends F to a larger field $F(i)$, then $O(3, F)$ is simply isomorphic with the group of all transformations

$$z' = \frac{\alpha z + \beta}{-\sigma \bar{\beta} z + \sigma \bar{\alpha}} \quad \left(\begin{array}{l} \alpha, \beta \text{ in } F(i), \\ \sigma \bar{\sigma} = 1. \end{array} \right).$$

The transformations with $\alpha \bar{\alpha} + \beta \bar{\beta} = 1$ form a subgroup, the fractional binary hyperorthogonal group (compare Linear Groups, page 132). Certain theorems on this group are established.

5. The second paper by Professor Dickson is a contribution to the theory of group matrices and group determinants, investigated by Frobenius and further developed by Burnside for continuous fields and by the writer for an arbitrary field. The paper employs only very elementary methods and is quite independent of the papers cited. For the cyclic group g_n , the dihedral group g_{2n} , the alternating and symmetric groups on four letters, and the alternating group on five letters, the group matrix is reduced to a simple canonical form of the type shown to exist by Frobenius, the reducing transformation being exhibited explicitly. In particular, the irreducible factors of the group determinant are given. Both papers (printed October 1) are to appear in the *University of Chicago Decennial Publications*, Volume 9.

6. Professor Hathaway contends that if a vector analysis short of quaternions is desired, then all attempts at bringing in conceptions of products of vectors should be avoided as necessarily incomplete and uselessly metaphysical. Any quantity whose value depends upon and is determined by the values of given arguments $\mathbf{x}\mathbf{y}\mathbf{z}$ is simply a function of those arguments, and the ordinary form of notation $f\mathbf{x}\mathbf{y}\mathbf{z}$ should suffice.

The requisite analysis is afforded by the linear functions, defined with respect to any argument $\mathbf{x}$, by

$$f(\mathbf{x} + \mathbf{x}')\mathbf{y}\mathbf{z} = f\mathbf{x}\mathbf{y}\mathbf{z} + f\mathbf{x}'\mathbf{y}\mathbf{z}.$$

In consequence, a scalar factor of an argument of a linear function is simply a factor of the function; and we have general expansions of the type

$$f(\mathbf{x} + \mathbf{x}')(\mathbf{y} + \mathbf{y}')\mathbf{z} = f\mathbf{x}\mathbf{y}\mathbf{z} + f\mathbf{x}\mathbf{y}'\mathbf{z} + f\mathbf{x}'\mathbf{y}\mathbf{z} + f\mathbf{x}'\mathbf{y}'\mathbf{z}.$$

The similarity of this to distributive multiplication is evident; and any symmetric or alternate properties of the function concerned impose corresponding commutative laws. Only three linear functions (a *scalar* and *vector* alternate, and a *scalar* symmetric) need be permanently defined:

$Sabc$ = volume of parallelepiped whose edges are abc in order.

Vab = vector of the area of the parallelogram whose edges are ab in order.

Sab = volume of parallelepiped of edge a and vector base area b .

7. In Professor Shaw's paper it was shown that the general form of any number of a nilpotent algebra is

$$\phi = \sum a_{ijk} \lambda_{ijk},$$

where $\lambda_{ijk} \lambda_{j'ik'} = \lambda_{iik+k'}$, $\lambda_{ijk} \lambda_{j'ik'} = 0 (j \neq j')$, and in any case λ_{rat} does not exist unless t is positive or zero, and $\mu_r - \mu_s - 1 < t < \mu_r$. The numbers μ_r , μ_s , etc., are certain "multiplicities" in the expressions. The units of the algebra may be expressed in the form

$$\phi_1, \phi_2, \dots, \phi_m, \phi_{m+1}, \phi_1 \phi_{m+1}, \phi_1 \phi_{m+1}^2, \dots$$

$$\phi_2 \phi_{m+1}, \phi_2 \phi_{m+1}^2, \dots$$

where

.....

$$\phi_r \phi_s = a \phi_{s+1} + b \phi_{s+2} + \dots + f \phi_{m+1} + g \phi_1 \phi_{m+1} + \dots$$

Certain typical cases are given as examples.

8. Professor Campbell's paper is in abstract as follows: Let

$$\sum_{i=1}^4 \sum_{j=1}^4 a_{ij} y_i y_j \quad (a_{ij} = a_{ji})$$

be a homogeneous quadratic form in the solutions y_1, y_2, y_3, y_4

of the linear differential equation $y^{iv} + 6p_2y'' + 4p_3y' + p_4y = 0$. It can be written symbolically as

$$\left(\sum_{i=1}^{i=4} \alpha_i y_i\right)^2.$$

If successive derivatives be taken of this form and use be made of the fact that $y_i^{iv} = -6p_2y_i'' - 4p_3y_i' - p_4y_i$ ($i = 1, 2, 3, 4$), any derivative will be composed of terms, each of which consists of a constant or a rational integral function of the coefficients, of the given differential equation, and their derivatives, multiplied by the symbolic factors

$$\left(\sum_{i=1}^{i=4} \alpha_i y_i\right)^{r_0} \left(\sum_{i=1}^{i=4} \alpha_i y_i'\right)^{r_1} \left(\sum_{i=1}^{i=4} \alpha_i y_i''\right)^{r_2} \left(\sum_{i=1}^{i=4} \alpha_i y_i'''\right)^{r_3}$$

where $r_0 + r_1 + r_2 + r_3 = 2$. These factors may be kept in the order

$$\left(\sum_{i=1}^{i=4} \alpha_i y_i^{(j)}\right) \quad (j = 1, 2, 3),$$

and indicated by the exponents $[r_1, r_2, r_3]$. Then the derivative of the symbolic part of each term of any derivative will be given by the formula

$$\begin{aligned} \frac{d[r_1 r_2 r_3]}{dx} &= r_1[r_1 - 1, r_2 + 1, r_3] + r_2[r_1, r_2 - 1, r_3 + 1] \\ &+ (2 - r_1 - r_2 - r_3)[r_1 + 1, r_2, r_3] - 6p_2 r_3[r_1, r_2 + 1, r_3 - 1] \\ &- 4p_3 r_3[r_1 + 1, r_2, r_3 - 1] - p_4 r_3[r_1, r_2, r_3 - 1]. \end{aligned}$$

The following theorems are used in the investigations:

First. If there are g functions U_i , $i = 1, 2, \dots, g$ of an independent variable among which ν and only ν linearly independent relations exist, then

$$u = \sum_{i=1}^{i=g} a_i U_i,$$

a_i arbitrary constants, is the general solution of a linear differential equation of order $g - \nu$.

Second. The necessary and sufficient condition that the n functions of x , U_i ($i = 1, 2, \dots, n$) are linearly independent is that the determinant

$$\begin{vmatrix} u_1 & u_2 & \dots & u_n \\ u'_1 & u'_2 & \dots & u'_n \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-1)} & u_2^{(n-1)} & \dots & u_n^{(n-1)} \end{vmatrix}$$

vanishes.

Third. In any homogeneous quadratic relation $u = [000]$ that may exist in the solutions of the given differential equation, the expression $[200]$ cannot vanish.

Fourth. The relations $[000] = 0$, $[\overline{000}] = 0$, $\dots$ in the solutions of the given differential equation are linearly independent if $[200]$, $[\overline{200}]$, $\dots$ are linearly independent.

By successive differentiation of $u = [000]$ there result ten equations in the expressions $[r_1, r_2, r_3]$ whose coefficients are constant or rational integral functions of the coefficients of the differential equation and their derivatives, and whose right hand members are $u, u', u'', \dots, u^{ix}$. Arrange the terms so that the expressions appear in the order $[000], [100], [010], [200], [110], [101], [020], [011], [001], [002]$. Form the determinant Δ of the coefficients. Denote any determinant formed from the matrix derived from Δ by omitting the last n rows, $n = 1, 2, 3$, by the numbers indicating the columns chosen for the determinant. It is shown that:

The necessary and sufficient conditions that one and only one homogeneous quadratic relation exists in the solutions of the given differential equation are that Δ vanishes and (1235678910) does not vanish. The necessary and sufficient conditions that two and only two linearly independent homogeneous quadratic relations exist in the solutions of the given differential equation are that (1235678910) vanishes and not all (123478910) , (123578910) and (123678910) vanish. The necessary and sufficient conditions that three linearly independent homogeneous quadratic relations exist in the solutions of the given differential equation are that (123478910) , (123578910) and (123678910) vanish. There cannot be more than three linearly independent homogeneous quadratic relations in the solutions of the given differential equation.

9. The first part of Dr. Slocum's paper finds the form of the infinitesimal transformation U_a by which any given finite transformation T_a of a group with continuous parameters is generated when the equations defining this finite transformation are in their *non-canonical* form. For certain values of the parameters the transformation U_a may not be infinitesimal, in which case the corresponding finite transformation T_a is not generated by an infinitesimal transformation of the group.

Associated with each r -parameter structure is a certain determinant the constituents of which are functions of the r -parameters. If any system of values of the parameters can be found for which this determinant vanishes, the parameter group belonging to that type of structure is discontinuous.

A group may be continuous as a real group but discontinuous as a complex group. Consequently the idea, as developed by Lie, of the relation between the transitivity of the real and complex groups which have the same symbols of infinitesimal transformation, requires modification.

There being more types of structure possible for real groups than for complex groups, if we have given two structures which are of the same type, A , for complex groups, but which constitute distinct types, B and C , for real groups, it is shown by means of examples that one of the following cases may occur with respect to the continuity of the real groups of types B and C : 1° All real groups of both types B and C are continuous. 2° All real groups of both types B and C are discontinuous. 3° All real groups of one type, B , are continuous, and one or more, but not all, real groups of the other type, C , are discontinuous. 4° All real groups of one type, B , are continuous, and all real groups of the other type, C , are discontinuous.

10. Through Culmann's investigations in graphical statics it is known that the directions corresponding to the sections and the resultant stresses upon these sections at every point of a thin strained plate, form an involutoric pencil. This involution is hyperbolic or elliptic according as the plate is either subject to tensions and compressions, or to tensions or compressions only. Professor Emch briefly discusses these cases, establishes the equation of the stress ellipse and the differential equation of the tension and compression curves, and finally applies the results to the linear infinitesimal deformation of the plane.

11. Following a plan outlined by Professor White, in a paper read before the Society in September, 1902, an attempt is made by Mr. Wilson to investigate the number of independent relations that must exist among the coefficients in order that a quartic curve shall contain an infinity of triangles self-conjugate with respect to a given conic.

The binary quartic and conic were taken in the form

$$f_x^4 = A_x^2 a_x^2 \equiv 0, \quad \phi_x^2 = \alpha_x^2 \equiv 0.$$

The condition that two roots of the quartic shall separate two roots of the conic harmonically is

$$\{(aa)^2(A\alpha)^2\} \{(AB)^2(ab)^2(\alpha\beta)^4 - (\alpha\beta)^2(A\alpha)^4 \\ + 4(\alpha\beta)^2(aa)^2(A\alpha)^2(A\alpha)^2 - 4(aa)^4(A\alpha)^4\} = 0$$

(see Clebsch, Binäre Formen.) Reducing and expressing in terms of invariants of $f_x^4 = 0$ and $\phi_x^2 = 0$, this becomes

$$\bar{A}^3 - \frac{1}{2}j\Delta^3 - \frac{1}{2}iA\Delta^2 \equiv 0,$$

where i and j are invariants of $f_x^4 = 0$ alone, Δ of $\phi_x^2 = 0$, and $\bar{A}$ a simultaneous invariant of f and ϕ .

By bordering, the condition is obtained that a line $\mu_x = 0$ (ternary) shall satisfy the harmonic division. By considering the locus of the pole of $\mu_x = 0$, a 12th-ic is obtained which breaks down to a sextic $C_x \equiv 0$. The general sextic contains 28 coefficients, and hence the maximum number of conditions is 28.

The curves

$$f_x^2 = 4ax_2^3x_3 + 4bx_3^3x_1 + 4cx_1^3x_2 = 0, \quad \phi_x^2 = x_1^2 + x_2^2 + x_3^2 = 0$$

have been examined and show that the quartic above cannot contain infinitely many inscribed triangles self-conjugate with respect to a conic which has the triangle $x_1x_2x_3 = 0$ as a self-conjugate triangle.

The forms

$$f_x^2 = 6 \sum a_1 x_1^2 x_3^2 + 12 \sum b_1 x_1^2 x_2 x_3 = 0, \quad \phi_x^2 = x_1^2 + x_2^2 + x_3^2 = 0$$

have also been examined. Although the result is not in complete form, it has disclosed properties that promise to lend interest to investigations other than the one in hand.

12. The formulas of Euler (and Rodrigues), which give rationally in terms of three parameters the nine coefficients of an orthogonal transformation in space, are available for expressing the vertices of any polar triangle of the conic

$$x_1^2 + x_2^2 + x_3^2 = 0.$$

This gives a means of discussing curves of the second and third orders, at least, which contain infinitely many inscribed polar triangles of a conic. Professor White's preliminary communication exhibited the method as applied to conics. The further question was raised, what sort of curve is the locus of points whose coördinates are the eulerian parameters of polar triangles inscribed in a single conic, or of orthogonal transformations which rotate the axes through the surface of an orthogonal quadric cone.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILLINOIS.

SOME GROUPS IN LOGIC.

BY PROFESSOR E. W. DAVIS.

(Read before the Chicago Section of the American Mathematical Society,
January 2, 1903.)

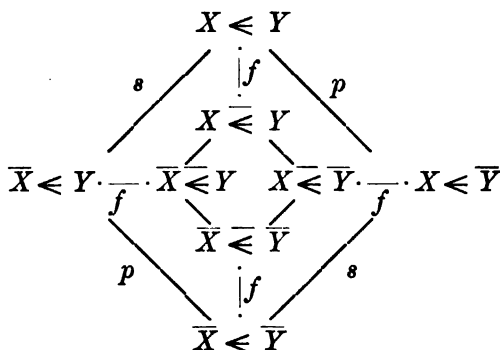
DE MORGAN has pointed out * that his eight forms of proposition identical with the *A*, *E*, *I*, *O*, and their contranominals of the older logic, can be derived from any one by the three operations of reversing the subject, reversing the predicate, denying the copula. If, in fact, we denote the operations in question by *s*, *p*, and *f* respectively, we have

$$Ap = E, \quad Af = O, \quad Afp = I;$$

while *sp* changes any proposition to its contranominal $X \leq Y$ to $Y \leq X$; or, what is the same thing to $\bar{X} \leq \bar{Y}$. Here $\leq$ is the sign of implication and the bar written over a letter or sym-

* Formal Logic, p. 63 et seq.

bol reverses or denies the meaning of the same. All the relations can be simply expressed by a diagram



This scheme differs from De Morgan's in that he assumes always the existence of both a subject and its contrary, whereas our assumption, following Mr. C. S. Peirce,* is that $X < Y$ means 'not X or else Y ,' while $X <= Y$ means ' X but not Y .'

There is also a group of order 32 which operating upon any one of 24 forms of valid syllogism will produce the rest, together with 8 non-valid syllogisms. To see this, start with *Barbara*

$$(X < Y)(Y < Z) < (X < Z),$$

or say

$$A_1 A_2 < A_3.$$

Let now a_1 mean the operation of performing *sfp* upon A_2 and A_3 , with similar significations for a_2 and a_3 . Then, B standing for *Barbara*, Ba_1 and Ba_2 , are valid, while Ba_3 is not.† Moreover

$$G_4 = \{1, a_1, a_2, a_3\}$$

is a 4-group. Were B , instead of *Barbara*, any valid syllogism it would still be true that of Ba_1 , Ba_2 , and Ba_3 , two would be valid while one was invalid. If B were invalid the operations a might lead to none that were valid, could at most lead to one that was.

* *Amer. Journ. of Math.*, vol. 3, p. 18.

† Conversion of the propositions with negative copula reveals the fact that B , Ba_1 , Ba_2 , Ba_3 , are in the four figures of the older logic.

Suppose x to be the operation of interchanging X and $\bar{X}$, with like significations for y and z . These 3 operations generate a group G_8 , such that when performed upon any syllogism B the resulting 8 syllogisms are valid or invalid according as B is.

The group is commutative with G_4 and the product of the two is the G_{32} referred to above.

Let s, p, f be the operations of performing s, p, f upon A_i . It is evident that

$$x = s_1 s_3, \quad y = p_1 s_2, \quad z = p_2 p_3, \quad a_1 = s_2 p_2 f_2 s_3 p_3 f_3.$$

These 9 operations generate a group G_{512} , the product of G_{32} by the group

$$G_{16} = \{x, s_1, p_1, p_2, f_1\}.$$

The new syllogisms got by operating with this upon the 32 syllogisms derived from *Barbara* above are all invalid.

The operation called conversion interchanges the subject and predicate of a proposition, at the same time reversing each. The meaning is, of course, left unchanged. Immaterial to the argument likewise is it whether the order of the propositions be $A_1 A_2$ or $A_2 A_1$. Let the operation of converting A_i be denoted by c_i while the interchange of A_1 and A_2 is denoted by t . Then $\{c_1, c_2, c_3, t\}$ is of order 16. We call it H_{16} . Finally

$$G_{512} \times H_{16} = G_{8192},$$

whereof

$$G_{32} \times H_{16} = H_{512}$$

contains the operations which performed upon *Barbara* lead to all the valid syllogisms of BG_{8192} taking account of conversion and the order of the premises, viz., to 384.

The same sort of group building can of course be applied to more complicated sets of statements, and is a very simple consequence of the binary character of deductive logic.

UNIVERSITY OF NEBRASKA,
January 11, 1903.

CESÀRO'S INTRINSIC GEOMETRY.

Lezioni di Geometria Intrinseca. By ERNESTO CESÀRO, professor of mathematics at the Royal University of Naples. Published by the author, Naples, 1896. 8vo., 264 pp., 48 figures.

Ernesto Cesàro, Vorlesungen über natürliche Geometrie. Authorized German translation by Dr. GERHARD KOWALEWSKI, docent at the University of Leipzig. Teubner, 1901. 8vo., vi + 341 pp., 48 figures.

THE subject of intrinsic geometry is almost unknown among American mathematicians, yet it has quite an extensive literature, which has been recently collected and published by Wölffing.* The name Cesàro occurs more frequently than any other in this report.

The work here under consideration contains considerable new matter, but its principal mission is to provide a systematic treatment of the subject from the beginning. The success of the effort is attested to by the fact that the work received honorable mention at the awarding of the Lobachevsky prize in 1897.

The German translation differs but little from the Italian original, except that it corrects a considerable number of typographical and other errors, and embodies in the text a number of notes appended to the original. The further changes will be mentioned later.

The work is divided into seventeen chapters, of which eight treat of plane geometry, two of twisted curves, three of surfaces, one of line congruences, and finally the last three of three-dimensional space, of curves in hyperspace, and of hyperspaces, respectively. There is also an appendix giving a generalization of Grassmann's numbers, a discussion of the equilibrium of flexible but non-extensible wires, and the equations of elasticity in hyperspace.

The physical improvements in the German edition, the spaced paragraphing, the full index (wholly lacking in the original)

* Bericht über den gegenwärtigen Stand der Lehre von den natürlichen Coördinaten, *Bibliotheca Mathematica*, Ser. 3, vol. 1, pp. 142-159.

and the more distinctive type used for printing the illustrative examples, are all of great assistance to the reader.

The subject matter proper of the book is not preceded by an introduction, and no references are given, both of which omissions greatly limit the usefulness of the work. The lack has now to a large extent been supplied, so far as the plane is concerned, by Wölffing's report mentioned above.

In the first five chapters a curve is defined by a relation between s , the length of arc measured from any point of the curve and ρ , the radius of curvature. Frequently rectangular coördinates (u, v) referred to the tangent and normal are used. The first chapter gives a good sample of the methods employed, by deriving a large number of interesting properties of certain curves in a brief and strikingly elegant manner.

Let $M(o, o)$ and $M'(u, v)$ be two points on a curve. Since $\sqrt{u^2 + v^2} < \delta s < u + v$, and $\lim v/u = 0$, it follows that $\lim \delta s/u = 1$. If MM' makes an angle $\delta\phi$ with the tangent at M , $1/\rho = \lim \delta\phi/\delta s$, $\lim v/u^2 = 1/\rho$. These ideas suffice to obtain the forms of the circle $\rho = a$, the catenary $\rho = a + s^2/a$, and the catenary of uniform resistance $\rho = \frac{1}{2}a(e^{-s/a} + e^{s/a})$. It is to be observed that the origin and axes are variable and are only intermediate steps in the analysis.

Inflections and cusps appear according as

$$\lim \frac{v}{u^{2-n}} = \frac{1}{(2-n)(1-n)} \lim \frac{s^n}{\rho}$$

is less or greater than zero. Asymptotes are characterized by $s = \infty$ when $1/\rho = 0$ and $\lim \phi$ is finite. Asymptotic circles result when $\lim \rho = a$, as $\phi = \infty$, and asymptotic points appear when $a = 0$. The study of these points closes the chapter.

The examples discuss the involute of the circle $\rho^2 = 2as$, the form of the tractrix, the proof that the segment of the tangent cut off by the asymptote is constant, and a simple construction for the center of curvature of any point. The German edition compares the forms of the two catenaries, and shows that the catenary of uniform resistance has two parallel asymptotes. The cycloids are defined by the equation

$$\frac{s^2}{a^2} + \frac{\rho^2}{b^2} = 1;$$

the hypocycloid or epicycloid is defined, according as a is

greater or less than b , and the common cycloid is the intermediate form, when $a = b$. The whole discussion, including eight figures, occupies less than two pages. Its directness must be admitted, but the periodic form of the curve is not established until an auxiliary variable is employed. From the form of the equation, no direct knowledge of the form of the curve beyond a cusp is given. The author makes no further mention of the matter, but ignores the fact that he departs from his otherwise consistent procedure to establish the result. This departure is, however, unnecessary if account be taken of the change of sign which the arc undergoes when passing a cusp. The same scheme will apply to all the curves which have cusps, whether periodic or not. This excellent list of examples closes with a discussion of the curves

$$\rho = ks^n,$$

which include the involute of the circle ($n = \frac{1}{2}$), the logarithmic spiral ($n = 1$) and the double spiral ($n = 2$) which Cesàro calls the clotoid.*

The second chapter discusses the fundamental formulas

$$\frac{dx}{ds} = \frac{y}{\rho} - 1, \quad \frac{dy}{ds} = -\frac{x}{\rho},$$

which are necessary and sufficient for the immovability of the point (x, y) not on the curve. These formulas are all that are needed for expressing the equation of any locus which is invariantly connected with the curve. Here envelopes, evolutes, parallel curves are discussed and a rich collection of examples is given which illustrate how to derive the intrinsic equations of curves defined as a geometric locus. The method is rather too abstract, and not sufficient use is made of coördinate axes. For example, it is proved that the evolute of a cycloid is a similar cycloid, but an indirect procedure is necessary to determine the relative positions of the two curves. The same fact is still more evident in case of the hypocycloid and epi-

* For a further discussion of the clotoid, and the literature upon it, see Loria's recent work, *Le curve piane algebriche e trascendenti, teoria e storia*; German by Schütte, *Specielle algebraische und transcendente Curven in der Ebene, Theorie und Geschichte*, Leipzig, Teubner, 1901, p. 457. It would be interesting to compare the various double spirals with the stereographic projection of the logarithmic spiral.

cycloid. The relation between ρ_n , the radius of curvature of the n th evolute, is shown to satisfy the equation

$$\rho_n = \rho \frac{d\rho_{n-1}}{ds}.$$

The third chapter is devoted to the discussion of particular curves. It begins with a review of five pages on the ordinary properties of conics, derived by the well known methods, after which the equation between s and ρ is expressed by means of an elliptic integral of the second kind. The equation of the cassinian oval is expressed as a hyperelliptic integral of the first kind, for which $p = 3$. A family of curves is also discussed for which Wölffing has suggested the name Cesàro curves. They are defined by the following property: the radius of curvature at any point is proportional to the segment on the normal between the point of tangency and its intersection with the polar of the point of tangency with regard to a fixed circle. When the circle reduces to a point the curve becomes a sinuous spiral. When the circle becomes a straight line, the locus is called a Ribaucour curve. Although various curious properties are derived, the method does not seem adapted to the study of the general topology of the curves; no figures are given except of the well known forms. The German edition has added one more paragraph [§ 46] to this chapter, dealing with the investigation of those curves whose osculating circles, dilated from the point of contact, all belong to a linear complex of circles. The locus of centers is a Cesàro curve.

In the fourth and fifth chapters, contact and osculation, and roulettes, the power of the method again becomes apparent. The ordinary properties of contact, curvature, successive evolutes, maximum and minimum curvature, etc., are very skilfully derived. In the chapter on roulettes a more detailed study of the cycloid is given, and a number of elegant relations between various Cesàro curves are derived.

Thus far the procedure has been successive; a small number of formulas was first derived and a large number of consequences drawn from them. The remaining chapters (except the ninth and tenth) use many auxiliary coördinates.

Chapter VI deals with barycentric coördinates. It would grace any work on modern geometry, and the analysis seems to lend itself readily to the method in question. Centers and lines

of gravity are discussed at length and applied to a large number of curves. Chapter VII, barycentric analysis, is a direct continuation of the preceding. Equations of a line, pair of lines, conic section, tangent, normal, pole and polar, are derived in much the same manner as in the ordinary texts on trilinear coördinates. The chapter closes with a well written paragraph on the symmetric triangular curves and the anharmonic curves. The lack of references here seems particularly unfortunate, for it is of considerable value to compare the treatment here given with that of Laguerre, Halphen, Klein and Lie.

The last chapter in the part on plane geometry is devoted to pencils of plane curves. Let u be a function of position in the plane. If u changes by the amount du , when the point moves through the distance ds ,

$$\frac{du}{ds} = \cos \omega \frac{\partial u}{\partial s_1} + \sin \omega \frac{\partial u}{\partial s_2},$$

in which ds_1, ds_2 , represent orthogonal components of ds . The expression

$$\Delta u = \left(\frac{\partial u}{\partial s_1} \right)^2 + \left(\frac{\partial u}{\partial s_2} \right)^2$$

is called the first differential parameter. If the pencil $u \equiv \kappa$ consists of parallel curves, Δu is a function of u alone. The second differential parameter

$$\Delta^2 u = \left(\frac{\partial}{\partial s_1} + \frac{\partial \omega}{\partial s_1} \right) \frac{\partial u}{\partial s_1} + \left(\frac{\partial}{\partial s_2} + \frac{\partial \omega}{\partial s_2} \right) \frac{\partial u}{\partial s_2}$$

vanishes for an isothermal system, provided u is the isometric parameter. These considerations furnish a good introduction to curvilinear coördinates, their fundamental formulas, and the conditions for integrability which follow. Then are added the relations among curvilinear coördinates, which were first derived by Lamé and by Bonnet, but which are usually found in works on the theory of surfaces.

The remainder of the book is devoted to geometry of more than two dimensions. Chapter IX treats of twisted curves and ruled surfaces. In the first part pure intrinsic geometry again becomes prominent, but the digression on the straight line

and the application of the results to scrolls, developables, and associated surfaces makes a wide departure from the otherwise consistent development. One has the feeling that the arrangement is far from natural in the sense of each result suggesting the following one, and the deductions frequently leave several steps unnoticed or assumed. The proof of the theorem that a developable must be the envelope of tangents of a twisted curve is direct and elegant. In establishing Chasles's correlation between tangent planes and points of tangency on a scroll the two lines $g, g + \delta g$ are confused; the tangent plane should contain the generator through the point of tangency. The next chapter discusses some particular twisted curves. Spherical curves, helices, and conical loxodromes are treated at some length and a number of miscellaneous examples are added. A discussion of the Bertrand curves, defined by a linear relation between curvature and torsion, is followed by a study of the curves satisfying the quadratic relation

$$\frac{A}{\rho^2} + \frac{B}{r^2} + \frac{C}{\rho r} = \frac{P}{\rho} + \frac{Q}{r}.$$

Although the various properties mentioned are all derived, the method of deduction is so briefly outlined and explanations are so sparingly used that the text would prove to be almost impossible reading to any one not fairly familiar with the results of the ordinary theory.

Chapter XI, general theory of surfaces, is largely geometric in character and not so materially different from the treatment in the ordinary books, except that it is remarkably brief and direct. Most of us are not satisfied with these considerations as proofs, but they are of immense value when accompanied by the lively geometric interpretation. There are no natural coördinates here as in the case of curves, but both this treatment and the ordinary one would be greatly improved by drawing from each other. One interesting theorem is the determination of the curvature in the tangent plane in a hyperbolic point (theorem of Beltrami), as the theorems of Euler and Meusnier do not apply to this case. The result shows that the curvature of the section made by the tangent plane is two thirds that of the asymptotic line which it touches.

The next statement seems trivial; it says that two curves may touch each other and have the same osculating plane at

the point of contact without having the same curvature at that point.

Clairaut's theorem regarding geodesics on surfaces of revolution is very neatly proved.

The German edition adds a theorem regarding the curvature of a normal section of the circumscribing cylinder of a surface (curve of apparent contour).

A good discussion of surfaces of constant curvature is given, and figures of the various types are discussed. Asymptotic lines on the pseudosphere are derived and the form of the lines obtained. Applicability, minimum surfaces and the Weingarten surfaces are briefly treated.

These two chapters are very valuable when read in connection with the older treatment of the subject; but the outlook is by no means hopeful that the later method will prove sufficiently powerful to secure many new results from the theory of surfaces. The chapter closes with rather a full discussion of surfaces of the second order.

Chapter XIII, infinitesimal deformations of a surface, discusses the possible deformations and the consequent changes of curvature. It is skilfully proved that the mean curvature of a minimum surface must be everywhere zero, and that the curvature of any surface is not changed by deformation.

The next chapter is devoted to rectilinear congruences. It is a rapid review of the work of Kummer, which is so well presented in the work of Bianchi. A certain arbitrariness is here apparent and a strong suspicion is unavoidable that all the results were known beforehand; moreover, one feels that the present method is forced somewhat in reproducing those results. The results of Sturm on the configuration of the normals of a surface near a given point are added to the older theorems of Hamilton and Kummer, and some new relations among the derivatives are obtained analogous to those of Codazzi for surfaces. By means of these there is given a noteworthy proof of the theorem that the mean envelope of an isotropic congruence is a minimum surface. The second half of this chapter is considerably altered in the German edition.

Chapter XV treats of three dimensional space. The discussion is very similar to the introduction of curvilinear coördinates in the plane. All the tangent lines to a surface at a particular point lie in a plane. In any triply orthogonal system the curves of intersection are lines of curvature. Every

pencil of parallel surfaces belongs to a triple orthogonal system. Several paragraphs are devoted to the derivation of formulas analogous to Codazzi's.

Thus far the space is considered linear, so that the element of arc, referred to the tangents of the three orthogonal lines of curvature, is expressed in the form

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

If the space is not linear but has a constant curvature $1/R$, it is shown that the element of arc is defined by

$$ds = \frac{\sqrt{dq_1^2 + dq_2^2 + dq_3^2}}{1 + \frac{1}{4R^2}(q_1^2 + q_2^2 + q_3^2)}.$$

Apart from giving the impression of being rather overloaded, this chapter is very interesting and the power of the methods is made particularly apparent in all of the geometric part of it.

Chapter XVI treats of curves in space of n dimensions. The various normals are first defined, the conditions for orthogonality deduced, and then the analogons to Frenet's formulas are derived, which are called the fundamental formulas for the intrinsic analysis of curves in hyperspace.

In the German edition an article is added which is interesting because it discovers a fundamental difference in the nature of spaces having an even number of dimensions from those having an odd number. The locus of a point having first, second, ..., n th curvatures all constant is spherical or helical, according as n is even or odd.

The last few pages of this chapter are devoted to barycentric analysis of hyperspace, quite analogous to the projective system of Professor Stringham.

The last chapter preceding the appendix is on hyperspaces. The generalizations of the theorems of Euler and Dupin are first derived, then the formulas of Codazzi and the criteria for infinitesimal deformation. The volume closes with the remarkable theorem of Beez that a non-extensible hypersurface cannot be deformed.

The book represents an important and a beautiful work, and the reader's admiration is constantly excited for the author's versatility and power of assimilating the existing literature.

Two general defects seem to be patent, one in presentation, one in range. It is the author's aim to emphasize the force and directness of the analysis by having the results appear after a minimum of intermediate work, but this work has frequently been so abbreviated that the reading is at times extremely difficult. Again, the author was naturally ambitious to have his method apply to as large a range of subjects as possible, but its applicability seems but questionably successful when applied to projective properties.

For the differential analysis of curves and surfaces the work of Cesàro is certainly a powerful instrument and can be used with profit by every student of the subject.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
December, 1902.

GAUSS'S COLLECTED WORKS.

Carl Friedrich Gauss Werke. Achter Band. Herausgegeben von der königlichen Gesellschaft der Wissenschaften zu Göttingen. In Commission bei B. G. Teubner in Leipzig, 1900. 4to., 458 pp.

WITH the death of Ernst Schering, the venerable editor of the first six volumes of Gauss's collected works, it became incumbent on younger hands to continue the labor of studying, selecting and preparing for the press everything of interest that still remained in the great mass of manuscript that Gauss left behind. Professor Klein, with customary energy and dispatch, has made the necessary arrangements to bring Schering's labor to a prompt termination.

Four new volumes are planned. Professor Brendel, of Göttingen, is editor-in-chief, having the editorial supervision of the entire undertaking. He has also, in particular, the preparation of volume VII. This is to contain the final edition of the *Theoria motus* and Gauss's voluminous work on the theory of perturbation of the smaller planets, the theory of the moon, etc.

Volume VIII is the volume under review and will be spoken of later.

The contents of volume IX fall into two parts. One of these is to be devoted to mathematical physics, and is in charge of Professor Wiechert, director of the magnetic observatory in

Göttingen. The other part in charge of Professors Börsch and Krüger of the geodetic Centralinstitut in Potsdam, is to contain matter relating to geodesy.

Volume X will contain extracts from Gauss's very extensive scientific correspondence, also biographical notes. Finally a supplementary volume is planned to give a complete index of the preceding ten volumes.

In this connection it is interesting to learn that the movement inaugurated by Professor Klein to collect letters written by or to Gauss, copies of his lectures, scattered manuscripts, note books, in short anything and everything which will help posterity to understand and appreciate Gauss's genius, has been eminently successful. The so-called Gauss rooms in the observatory at Göttingen now serve to house Gauss's papers, and memorabilia relating to him. These, when finally arranged, will without doubt be made accessible to the public and form a fresh inducement, if indeed any were needed, for the student of mathematics to visit Göttingen.

We turn now to the consideration of volume VIII, the first of the new series to appear. Its contents may be regarded as addenda to the first three volumes of Schering's edition, and to the fourth, excluding geodesy.

The new material falls under the following heads : Arithmetic and algebra, pp. 3-32 ; Analysis and function theory, pp. 36-117 ; all under the charge of Professor Fricke, of Braunschweig. Numerical calculation, pp. 121-130 ; Calculus of probabilities, pp. 133-156, in care of Professors Börsch and Krüger, of Potsdam. Foundations of geometry, pp. 159-268 ; Geometria situs, pp. 271-286 ; Problems and theorems in elementary geometry, pp. 289-300 ; Application of complex numbers to geometry, pp. 303-362 ; Theory of surfaces, pp. 365-449, all under the care of Professor Stäckel, in Kiel, with the exception of a single paper on developable surfaces, three pages in length, by Professor Weingarten, of Charlottenburg.

As the reader may know, the first three volumes of Schering's edition, devoted chiefly to the theory of numbers and algebra, the elliptic and hypergeometric functions, contained a very large amount of material taken from Gauss's posthumous papers. It is not astonishing, then, that a second gleaning does not bring to light much new matter on these topics. Interesting however are the following. First we note two short fragments on cubic and biquadratic residues. In the two

epoch-making memoirs on biquadratic residues published in 1828 and 1832 Gauss took the momentous step of introducing imaginary numbers into the theory of numbers. In two places in these memoirs Gauss states that he began to study the higher laws of reciprocity in 1805 and was soon led to introduce the roots of unity into his speculations as the true basis of this theory. The above notes have now an especial interest to the devotees of the sublime science, as Gauss often styled the higher arithmetic, because they belong to this early epoch, 1805–1808. Let us observe in passing that the ideas contained in the memoir of 1832, properly generalized by Kummer, Dedekind and Kronecker, have led to the modern theory of algebraic numbers.

In the section devoted to Analysis and theory of functions, three papers attract our attention. One is the celebrated letter of 1811 to Bessel. Here Gauss discusses summarily the nature of a function and an integral taken between complex limits. In the first place, he advocates warmly the introduction of the complex variable into analysis. He urges: * “Es ist hier nicht von praktischem Nutzen die Rede, sondern die Analysis ist mir eine selbständige Wissenschaft die durch Zurücksetzung jener fingirten Grössen ausserordentlich an Schönheit und Rundung verlieren und alle Augenblick Wahrheiten, die sonst allgemein gelten, höchst lästige Beschränkungen beizufügen genötigt sein würde.”

After stating Cauchy's theorem that $\int f(x)dx$ around a closed contour containing no singular points is zero he remarks †: “Dies ist ein sehr schöner Lehrsatz dessen eben nicht schweren Beweis ich bei einer schicklichen Gelegenheit geben werde. Er hängt mit schönen andern Wahrheiten, die Entwicklungen in Reihen betreffend, zusammen.”

The latter part of this quotation suggests at once far-reaching consequences: Does Gauss have in mind the developments of Taylor, Laurent and Fourier? In this letter Gauss also emphasizes the importance of those functions which we call to-day one valued integral transcendental functions. In place of the function Soldner had introduced, viz., $\int (\log x)^{-1} dx$, he suggests $\int (e^x - 1)x^{-1} dx$.

* P. 90. We take this occasion to remark that all references without indication of the title refer to Gauss's Works, Vol. VIII.

† P. 91.

The second paper, which Fricke dates about 1800, relates to the inversion of the elliptic integral

$$u = \int \frac{dx}{\sqrt{(1-x^2)(1-\mu^2x^2)}}.$$

We find to our surprise that the development given by Gauss is nothing more nor less than Weierstrass's celebrated expression of x as the quotient of two integral transcendental functions, viz., $Al_1(u)$, $Al_0(u)$.

The third interesting feature in the section are three fragments on elliptic modular functions. If the mathematical world was filled with astonishment in 1827-28 when it became gradually known that Gauss had had in his possession for thirty years a good part of Abel's and Jacobi's results, we to-day are hardly less astonished to learn that Gauss certainly knew some of the modern theory of the elliptic modular functions. In fact, on page 104 we find for example a modular figure with the angles $\pi/4$, $\pi/4$, $\pi/4$, complete with its *orthogonal circle*. It is worthy of note that on page 478 of volume III appears another figure very familiar to students of the modular functions, viz., the fundamental domain of the modulus $-ik^2$. But as this volume was published in 1876, while Fuchs's and Dedekind's papers, which gave birth to the modular functions, did not appear till the following year, the figure last mentioned could not mean much to Schering, otherwise he would have searched the manuscripts of Gauss more carefully for other fragments of this nature.

The section devoted to the calculus of probabilities contains some interesting letters of Gauss. The theory of least squares, so important in the adjustment of observations, was discovered independently by Legendre and Gauss. The first published account of it is found in Legendre's *Nouvelles méthodes pour la détermination des orbites des comètes*, which appeared in 1806. In the *Theoria motus*, published in 1809, Gauss claimed to have discovered it in 1795. A controversy arose, very painful to the friends of Gauss. For example, Legendre in a letter* to Jacobi dated 1827, speaking of Gauss's claims of priority in the elliptic functions, breaks out: "Comment se fait-il que M. Gauss ait osé vous faire dire que la plupart de vos théorèmes lui était connus, et qu'il en avait fait la découverte dès 1808? * * * Cet

* Jacobi, Werke, Vol. I, p. 398.

excès d'impudence n'est pas croyable de la part d'un homme qui a assez de mérite personnel pour n'avoir pas besoin de s'approprier les découvertes des autres * * * Mais c'est le même homme qui en 1801 voulut s'attribuer la découverte de la loi de reciprocité publiée en 1785 et qui voulut s'emparer en 1809 de la méthode des moindre carrés publiée en 1805. D'autres exemples se trouveraient en d'autres lieux, mais un homme d'honneur doit se garder de les imiter."

Such direct accusations were of course not published, but they were circulated in private. Enough was said in print and in private to cause Gauss's friend Schumacher to write him (1831) as follows: * "Da Sie die Resultate Ihrer Rechnung geben, so scheint es mir, ist es leicht zu zeigen dass diese durch die Methode der kleinsten Quadrate abgeleitet sind. Zach lebt zudem noch, und hat gewiss Ihren Brief aufgehoben. Finden Sie es nicht der Mühe werth endlich die Sache einmal, selbst gegen die mir vor allen widerlichen *höflichen* Zweifel der Franzosen unwidersprechlich abzumachen"? To this Gauss replies: † "Ich glaube Ihnen schon einmal geschrieben zu haben, dass ich auf keinen Fall diese Stelle worin die Methode zum erstenmale öffentlich angedeutet ist releviren werde, auch nicht wünsche, dass einer meiner Freunde mit meiner Zustimmung es thue. Diess hiesse *anerkennen als bedürfe* meine Anzeige (Theoria Motus Corporum Coelestium) dass ich seit 1794 diese Methode vielfach gebraucht habe, einer Rechtfertigung, und dazu werde ich mich nie verstehen. Als Olbers attestirte dass ich ihm 1803 die ganze Methode mitgetheilt habe, war diess zwar gut gemeint: hätte er mich aber vorher gefragt, so würde ich es hautement gemissbilligt haben."

The grandeur and inflexibility of Gauss's nature is finely illustrated by this passage. In a later letter ‡ (1840) to Schumacher we find Gauss's own estimate of the importance of the discovery of least squares. He writes: "Sie wissen dass ich selbst auf das von mir seit 1794 gebrauchte Verfahren * * * niemals grossen Werth gelegt habe. Verstehen Sie mich recht; nicht in Beziehung auf den grossen Nutzen den sie leistet, der ist klar genug, aber *danach* taxire ich die Dinge nicht. Sondern deshalb oder in so fern legte ich nicht viel Werth darauf, als vom ersten Anfang an der Gedanke mir so

* P. 137.

† P. 138.

‡ P. 141.

natürlich, so äusserst nahe liegend schien dass ich nicht im Geringsten zweifelte, viele Personen, die mit Zahlenrechnung zu verkehren gehabt, müssten von selbst auf einen solchen Kunstgriff gekommen sein und ihn gebraucht haben ohne deswegen es der Mühe werth zu halten viel Aufhebens von einer so natürlichen Sache zu machen."

The precocious genius of Gauss will stand forth when we recall that he was but seventeen years old when he discovered his method of least squares.

Before passing on to the next section we must call attention to a curious mishap which has befallen the editors. Thirty of the hundred pages given to the section of analysis and theory of functions are occupied by a memoir entitled: "*De integratione formulae differentialis $(1 + n \cos \phi)^n d\phi$.*" The manuscript of this memoir was discovered in 1893 and published by Schering with explanatory notes in the *Göttinger Nachrichten*. Fricke in a note * remarks: "Es fehlt in der Urschrift der Name des Verfassers und eben so auch eine Angabe über Ort, und Zeit der Abfassung. In dessen beseitigte der Vergleich sonstiger Manuscripte von Gauss mit der vorliegenden Handschrift jeden Zweifel welcher sonst vielleicht an Gauss' Autorschaft bestehen könnte." Fricke places it in the year 1795 or before, i. e., before or at the beginning of Gauss's student days in Göttingen. It turns out now, however, that the memoir was not written by Gauss at all, but by a certain Thibaut, who presented it to the Academy of Göttingen in the year 1799.†

We come now to the most interesting section of the book, the Foundations of Geometry, or the Non-Euclidean Geometry. Here as in the case of the elliptic functions Gauss had explored far into a territory a quarter of a century or more in advance of his contemporaries and for years was in possession of epoch-making results. But he never published anything on this subject. In letters to a few of his intimate friends he would occasionally explain some of his views or indicate in a fragmentary manner some of his results. Also on several occasions he reviewed in the *Göttinger Anzeigen* works touching on the theory of parallels. All that is known of Gauss's views relating to the non-euclidean geometry has been collected and put in this section.

* P. 64.

† F. Klein *Göttinger Nachrichten (Geschäftliche Mitth.)*, 1902, p. 12.

Gauss's reason for not publishing his results will be made clear from the following extracts. In a letter* to his student friend, W. Bolyai (1799), he writes: "Mach' doch ja Deine Arbeit bald bekannt; gewiss wirst Du dafür den Dank nicht zwar des grossen Publicums (worunter auch mancher gehört, der für einen geschickten Mathematiker gehalten wird) einern, denn ich überzeuge mich immer mehr dass die Zahl wahrer Geometer äusserst gering ist, und die meisten die Schwierigkeiten bei solchen Arbeiten weder beurtheilen noch selbst einmal sie verstehen können — aber gewiss den Dank aller derer, deren Urtheil Dir allein wirklich schätzbar sein kann."

In 1818 Gerling writes† Gauss that he intends to prepare a new edition of Lorenz's *Reine Mathematik* and wishes to know Gauss's opinion "wie es mit der Parallelentheorie wohl am besten zu halten ist." Gauss replies:‡ "Ich freue mich, dass Sie den Muth haben sich so auszudrücken, als wenn Sie die Möglichkeit, dass unsere Parallelentheorie, mithin unsere ganze Geometrie, falsch wäre, anerkennen. Aber die Wespen, deren Nest Sie aufstören werden Ihnen um den Kopf fliegen."

In a letter to Bessel (1829) Gauss writes:§ "Auch hier habe ich manches noch weiter consolidirt und meine Überzeugung dass wir die Geometrie nicht vollständig a priori begründen können, ist, wo möglich, noch fester geworden. Inzwischen werde ich wohl noch lange nicht dazu kommen meine *sehr ausgedehnten* Untersuchungen darüber zur öffentlichen Bekanntmachung auszuarbeiten, und vielleicht wird dies auch bei meinen Lebzeiten nie geschehen, da ich das Geschrei der Böötier scheue, wenn ich meine Ansicht *ganz* aussprechen wollte."

The wasps and the Boeotians that Gauss here refers to are probably not so much the mathematicians as the philosophers. As will be recalled, Kant in his *Kritik der reinen Vernunft* (1781) had endeavored to show that our notions of space were *essentially* a priori, *i. e.*, independent of all experience. In fact his demonstration of the possibility of synthetic judgments a priori, one of the corner stones of his philosophy,|| depends largely on this view of space.

* P. 159.

† P. 178.

‡ P. 179.

§ P. 200.

|| "Auf der Auflösung dieser Aufgabe oder einem genuthuenden Beweise, dass die Möglichkeit die sie erklärt zu wissen verlangt in der That gar nicht stattfinden, beruht nun das Stehen und Fallen der Metaphysik." *Kritik der reinen Vernunft*, ed. B. Erdmann (1884), p. 41.

Gauss's opinion regarding the *a priori* character of our notions of space is perhaps seen best from the following extract of a letter to Bessel (1830). He writes :* "Wahre Freude hat mir die Leichtigkeit gemacht, mit der Sie in meine Ansichten über die Geometrie eingegangen sind, zumal da so wenige offenen Sinn dafür haben. Nach meiner innigsten Überzeugung hat die Raumlehre in unserem Wissen *a priori* eine ganz andere Stellung wie die reine Grössenlehre; es geht unserer Kenntniss von jener durchaus diejenige vollständige Überzeugung von ihrer Notwendigkeit (also auch von ihrer absoluten Wahrheit) ab die der letzern eigen ist; wir müssen in Demuth zugeben dass, wenn die Zahl *bloss* unsers Geistes Product ist, der Raum auch ausser unserm Geiste eine Realität hat der wir *a priori* ihre Gesetze nicht vollständig vorschreiben können."

His scorn for the philosophers is manifested in a letter to Taurinus (1824); he writes,† speaking of some of the peculiarities of the non-euclidean geometry: "Aber mir deucht, wir wissen, trotz der nichtssagenden Wort-Weisheit der Metaphysiker eigentlich zu wenig oder gar nichts über das wahre Wesen des Raumes, als dass wir etwas uns unnatürlich vorkommendes mit *Absolut Unmöglich* verwechseln dürfen."

Again in a letter to Schumacher (1846), speaking of the notion of right and left, he writes: ‡ "Welche Geltung diese Sache in der Metaphysik hat, und dass ich darin die schlagende Widerlegung von Kants Einbildung finde, der Raum sei *bloss* die Form unserer äussern Anschauung habe ich succinct in den Göttingischen Gelehrten Anzeigen 1831, S. 635 ausgesprochen."

We have expressed our astonishment at Gauss's reluctance to prepare for publication the results of his speculations. We now wish to cite a couple of passages which exhibit another trait of his character, the utter lack of envy he displays when his results have been discovered by others and the carelessness he constantly manifests in establishing his rights of priority.

In a letter (1804) to W. Bolyai,§ speaking of his and Bolyai's investigations in non-euclidean geometry, he writes: "Ich habe zwar noch immer die Hoffnung dass jene Klippen einst und noch vor meinem Ende, eine Durchfahrt erlauben werden. Indess habe ich jetzt so manche andere Beschäftigungen vor

* P. 201.

† P. 187.

‡ P. 247.

§ P. 160.

der Hand dass ich gegenwärtig daran nicht denken kann, und glaube mir es soll mich herzlich freuen wenn Du mir zuvorkommst, und es Dir gelingt alle Hindernisse zu übersteigen. Ich würde dann mit der innigsten Freude alles thun, um Dein Verdienst gelten zu machen und ins Licht zu stellen so viel in meinen Kräften steht."

Again on the same subject in a letter to Gerling (1819) he writes: * "Die Notiz von Herrn Professor Schweikart hat mir ungemein viel Vergnügen gemacht und ich bitte ihn darüber von mir recht viel Schönes zu sagen. Es ist mir fast alles aus der Seele geschrieben."

Finally, the following extract † from a letter to W. Bolyai (1830): "Jetzt Einiges über die Arbeit Deines Sohnes. Wenn ich damit anfangen *'dass ich solche nicht loben darf'*: so wirst Du wohl einen Augenblick stutzen: aber ich kann nicht anders; sie loben hiesse mich selbst loben: denn der ganze Inhalt, der Weg, den Dein Sohn eingeschlagen hat, und die Resultate, zu denen er geführt ist, kommen fast durchgehends mit meinen eigenen, zum Theile seit 30–35 Jahren angestellten Meditationen überein. In der That bin ich dadurch auf das Aeusserste überrascht. Mein Vorsatz war von meiner eigenen Arbeit, von der übrigens bis jetzt wenig zu Papier gebracht war, bei meinen Lebzeiten gar nichts bekannt werden zu lassen. Die meisten Menschen haben gar nicht den rechten Sinn für das, worauf es dabei ankommt, und ich habe nur wenige Menschen gefunden, die das, was ich ihnen mittheilte, mit besonderm Interesse aufnahmen. Um das zu können, muss man erst recht lebendig gefühlt haben, was eigentlich fehlt, und darüber sind die meisten Menschen ganz unklar. Dagegen war meine Absicht, mit der Zeit alles so zu Papier zu bringen, dass es wenigstens mit mir dereinst nicht unterginge. Sehr bin ich also überrascht, dass diese Bemühung mir nun erspart werden kann, und höchst erfreulich ist es mir dass gerade der Sohn meines alten Freundes es ist der mir auf eine so merkwürdige Art zuvorgekommen ist."

Before leaving this section we must mention two fragments which give rise to bold surmises. We recall that the advance that Riemann made in his Habilitationsschrift (1854) "Über die Hypothesen welche der Geometrie zu Grunde liegen" ‡ was

* P. 181.

† P. 220.

‡ Riemann's Werke, p. 254, et.

due in part to his introduction of differential geometry into his investigations, whereas the Bolyais, Lobachevsky and Gauss (certainly in his early work) employed the synthetic methods of Euclid.

Now in one of the above mentioned fragments Gauss treats this problem: Assuming that the geometry of Euclid is valid for infinitely small triangles, determine the relations between the sides and the angles of finite triangles. This leads to a differential equation whose constant of integration Gauss represents by k , which is precisely the letter he employs in his *Disquisitiones generales circa superficies curvas* to represent the measure of curvature. In the second fragment we have a deduction of the equations of the pseudosphere which Gauss calls "das Gegenstück der Kugel." There is in these two fragments no direct evidence that Gauss saw that his non-euclidean geometry found in the pseudosphere a geometric substratum, nor that he saw the intimate relation between the non-euclidean geometry and his theory of developable surfaces; but the supposition lies very near and is not improbable.

We pass over the intermediate sections in order to employ the remaining space at our disposal to touch on a topic of great interest to the English-speaking race, viz., Gauss's relation to linear associative algebra in general and to quaternions in particular.

We are all familiar with Hamilton's efforts, extending over a period of fifteen years, to discover a system of numbers in three units which would bear the same relation to space as the ordinary complex numbers $x + iy$ do to the plane, and which at the same time obey the formal laws of ordinary algebra. Finally in 1843 he discovered the quaternions. Meantime, Gauss had considered the same problem. In his report in the *Göttinger Anzeigen* (1831) of his memoirs on biquadratic residues which we have already referred to, Gauss after discussing the nature of the complex numbers $x + iy$ in a masterly manner of closes his remarks with the following oracular statement: * "Wir haben geglaubt, den Freunden der Mathematik durch diese kurze Darstellung der Hauptmomente einer neuen Theorie der sogenannten imaginären Grössen einen Dienst zu erweisen. Hat man diesen Gegenstand bisher aus einem falschen Gesichtspunkt betrachtet, und eine geheimnissvolle Dunkelheit dabei gefunden, so ist diess grösstentheils den wenig

* Volume II, pp. 177.

schicklichen Benennungen zuzuschreiben. * * * Der Verfasser hat sich vorbehalten den Gegenstand welcher in der vorliegenden Abhandlung eigentlich nur gelegentlich berührt ist, künftig vollständiger zu bearbeiten, wo dann auch die Frage warum die Relationen zwischen Dingen die eine Mannigfaltigkeit von mehr als zwei Dimensionen darbieten, nicht noch andere in der allgemein Arithmetik zulässige Arten von Grössen liefern können, ihre Beantwortung finden wird."

From this passage we see that Gauss already in 1831 had recognized the non-existence of a number system such as Hamilton so long vainly sought to find. The question at once suggests itself: If Gauss was able to state a theorem of such fundamental importance in the theory of higher complex number systems, how far did his researches reach, what was their nature? The new volume of the Werke, while it still leaves us entirely in the dark on this subject, brings however one great surprise. In a fragment* entitled Transformationen des Raumes we find the multiplication table of quaternions. This note is dated by Stäckel, 1819. That the multiplication is not commutative is noted page 360 and the relation between quaternions and spherical triangles is noted pages 360-361. To prevent a possible disappointment to any one referring to these passages we observe that Gauss does not employ the usual quaternion notation $a + bi + cj + dk$ but the earlier notation of Hamilton (a, b, c, d). The introduction of certain symbols for the units $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ is entirely unnecessary and is made merely for convenience. From a theoretical standpoint the subject even wins in clearness by leaving them out at the start.

Apropos of quaternions we cannot resist the temptation to cite a passage from a letter † to Schumacher (1843) relating to Möbius's Barycentric Calculus. What Gauss here says of the barycentric calculus can be applied directly to quaternions and the vector analysis, a subject dear to many English and American mathematicians. After mentioning the prejudice he first had toward Möbius's work and his doubt "ob es der Mühe werth sei, eine recht artig ausgesonnene Rechnungsweise sich anzueignen, wenn man durch dieselbe nichts leisten könne was sich nicht eben so leicht ohne sie leisten lasse," follows this remarkable passage which Hamilton and Gibbs would

* P. 358.

† P. 297.

doubtless subscribe to most cordially : " Ueberhaupt verhält es sich mit allen solchen neuen Calculs so, dass man durch sie nichts leisten kann, was nicht ohne sie zu leisten wäre ; der Vortheil ist aber der, dass, wenn ein solcher Calcul dem innersten Wesen vielfach vorkommender Bedürfnisse correspondirt, jeder, der sich ihn ganz angeeignet hat, auch ohne die gleichsam unbewussten Inspirationen des Genies die niemand erzwingen kann, die dahin gehörigen Aufgaben lösen, ja selbst in so verwickelten Fällen gleichsam mechanisch lösen kann, wo ohne eine solche Hülfe auch das Genie ohnmächtig wird. * * * Es werden durch solche Conceptionen unzählige Aufgaben, die sonst vereinzelt stehen, und jedesmal neue Efforts (kleinere oder grössere) des Erfindungsgeistes erfordern, gleichsam zu einem organischen Reiche."

In closing, we wish to praise the care the editors have taken in preparing the contents of the present volume for publication. As far as possible the dates of the various papers have been ascertained. Obscure passages have been explained and errors either corrected or pointed out. There are instances where the editors have been too sparing with their commentary, but they have doubtless preferred to err on the side of brevity rather than diffuseness.

Only one oversight has been noticed by the reviewer. In the extract printed on page 247, Gauss makes reference to his "schlagende Widerlegung" of Kant's theory of space which appeared in the *Göttinger Anzeigen* for 1831. This is certainly a matter of some importance and a reference should have been given where readers who have not the files of the *Anzeigen* at hand—and their number is relatively small—can find it in the *Werke*. The passage Gauss has in mind is to be found in volume 2, page 177 and footnote of the *Werke*. In this connection we wish to express the hope that further volumes will give *exact* references to the places where each particular paper was first published. This is only partially done in the volumes which have so far appeared. Indeed it seems desirable that this deficiency in the early volumes be made good in the supplementary volume.

In regard to misprints, we have observed but one, viz., page 160, last line : omit one *herzlich*. *why? perhaps in the original?*

To sum up, we may say that the volume under review takes its place worthily besides the six preceding volumes which

enjoyed the loving attention of one who sacrificed a large part of his scientific activity in erecting a lasting monument to the memory of Germany's princeps mathematicorum.

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ANALYTIC PROJECTIVE GEOMETRY.

Lehrbuch der analytischen Geometrie in homogenen Koordinaten.

Von WILHELM KILLING. 2 Teile, 8vo. Paderborn, F. Schöningh. Teil 1: *Die ebene Geometrie*, 1900. xiii + 220 pp. Teil 2: *Die Geometrie des Raumes*, 1901. viii + 361 pp.

THE first and perhaps the most noticeable feature of Professor Killing's text-book on the analytic geometry of homogeneous coördinates is the elaborately methodical arrangement. The line of march has been laid out with extreme care. Notice, for example, the titles and page numbers of the first few sections of the two volumes :

In the plane, Vol. 1.	In space, Vol. 2.
§ 1. Theory of cross ratio, p. 1.	§ 1. Division of dihedral angles, p. 1.
§ 2. The coördinate triangle, p. 8.	§ 2. The coördinate tetrahedron, p. 7.
§ 3. The straight line, p. 13.	§ 3. The straight line and plane, p. 13.
§ 4. The perpendiculars dropped upon a straight line from the vertices of the coördinate triangle, p. 19.	§ 4. The perpendiculars dropped upon a plane from the vertices of the coördinate tetrahedron, p. 21.
§ 5. The most general trimetric coördinates, p. 25.	§ 5. The most general tetrahedral coördinates, p. 29.
§ 6. The ratios of the coördinates as cross ratios, p. 37.	§ 6. The elements at infinity, p. 38.
§ 8. The elements at infinity, p. 48.	§ 7. The quotients of the coördinates as cross ratios, p. 44.
Total number of pages, 49.	Total number of pages, 52.

There is no necessity of stopping the parallel column here. It might run, with very few exceptions, to the end of the two volumes. Each closes with the treatment of confocal quadrics; the treatment in space is, of course, more elaborate than that required in the plane, but otherwise exactly parallel to it. The differences between the texts of Volume 1 and Volume 2 in the first fifty pages are of about the same order as the difference in the sectional headings quoted above. The changes are of two kinds: Those that are necessitated by the subject — the substitution of "line and plane" for "line," of four variables or four equations for three; and those changes which seem unnecessary — the substitution of the word "quotient" for "ratio" in the sectional headings 6 and 7, and the rearrangement of the order of the sections where it is of no particular importance.

This is carrying method to a painful extreme. There can be little excuse for spending 52 pages of Volume 2 upon those sections which correspond exactly to the 49 pages of Volume 1. The author might, at least, have economized in Volume 2 enough to save the three extra pages needed by the constantly recurring fourth variable and fourth equation. One could throw away the first part and commence on the second without experiencing any particular difficulty. So much elaboration of method is quite unnecessary. The book is well named a *Lehrbuch*, and one can almost hear, almost see the *Lehrmeister* as he reaches the second part say to his pupils: "I have explained the first volume with all care. Here is the second. Read it. It is like the first volume. Toward the end there are a few new things — no, not new, only a little less familiar, a little harder to generalize. If you find difficulty, wake me." And it would be a worthless student who need wake his master. More likely he would himself fall asleep over the monotony, the lack of opportunities to employ his wits and imagination.

Apart from this feature and from its antiquated typography, with spaced letters instead of italics and formulæ in the same font as the text, the book is good. The general plan is excellent; the scope has been chosen with judgment. The subjects treated are not too many nor too few. The methods of analysis are for the most part neat. The author has exercised great care in treating the matter of sign, which is so confusing to the beginner in connection with the coördinate triangle. Although the book is confessedly analytic there is often a glimmer, unfortu-

nately only a glimmer, of truly geometric reasoning. The large numbers of exercises distributed through the text, some at the end of each section, have been selected with especial care, assorted, and graded. For this reason the book must be more than usually helpful to a teacher, especially a young and inexperienced teacher, in preparing a course on projective geometry. To a student, who works without a teacher, who wishes to supplement a meager knowledge or even to gain from the beginning some idea of projective geometry, this book with its careful explanations and its problems will prove a great boon. It offers but little difficulty. It is so easy, elementary and slow; but withal, when one has finished it, so inclusive.

A comparison of the book under review with Duporcq's *Premiers principes de géométrie moderne* * is very instructive and interesting. Both treat projective geometry, though Duporcq concludes with a chapter on other transformations. But the methods are entirely different. The one is methodical, the other not at all; the one painstakingly accurate in demonstration, the other often hasty and fond of relying upon the sweep of intuition; the one analytic, the other geometric with only a background of analysis, like Chasles; the one assuring, the other stimulating. The fact is that a student of Killing would know that he had at his command a perfectly definite method. On meeting a problem he could work out a solution and probably put it in good form. Yet he very likely would not be one whit a geometer, but merely an analyst. The student of Duporcq could hardly get along without a teacher to help him. He would be quite unable to say whether he had a method at his command or not. He would know many theorems, and if he came to a problem he might or might not solve it. If he solved it he would not know why he had succeeded, nor in what section of geometry to place it. Yet he would think geometrically. The two books are complements of each other. The student of either would find in the other just what he lacked. If he were studying alone, he had best begin with Killing; if with a teacher, with Duporcq.

The place of trilinear and tetrahedral coördinates, the place of projective geometry in general in our mathematical education, has changed considerably since the first appearance of

* Reviewed in the BULLETIN, vol. 6, p. 254.

those classics — Salmon's Conic Sections and Clebsch's *Vorlesungen über Geometrie* — which, with their various translations into other languages, have done so much to influence the subsequent treatment of projective geometry. Thirty or forty years ago the group of projective transformations, then scarcely recognized as a group, was playing a well-nigh all-important rôle in mathematics owing to its connection with the so-called higher algebra. The vast theory of invariants — invariants of the projective group — was in building. Cayley and Clebsch were doing their great work. The particular enthusiasm of that day has passed by. As one looks over the announcements of the different universities he looks almost in vain for a course upon this theory of invariants. The theory of invariants now means a part of Lie's theory of groups.

The projective transformation itself seems less important. Inversion with its applications to the theory of functions, to potential, to all problems of conformality demands constantly more attention. Moreover there is the vast body of transformations brought to our notice by Lie under the name of contact transformations, of which even a single one, the line-sphere transformation, is of great importance. Thus even if we look only at the field of geometry we must grant that the projective transformation cannot hope to occupy so much of one's time and thoughts as formerly. And when we take into consideration other fields which have been springing up so rapidly and which contain scarcely the slightest reference to projective geometry, we may well wonder whether or not this subject deserves any special consideration, whether we are not giving it too much attention, whether in reality we need it at all except as a simple example of a group of transformations with very interesting properties.

Certain it is that the elaborate volumes on trilinear and tetrahedral coördinates are less attractive now. Room and time must be left for circle and sphere and line (in space) coördinates. Certain it is, too, that now, after the passing of the formal invariant theory of the projective group, projective geometry is taking its place beside other important geometries and we must in every way try to present its essentials as rapidly as possible and without any hope of elaborating its details. The other geometries cannot longer be neglected.

That, however, which will always render projective geometry the best introduction to other geometries is its geometric sim-

plicity and elegance. Its conceptions, its transformations are more easily visualized. The figures with which it deals are of the simplest. The methods of treatment have been so developed that one may proceed analytically or synthetically and may thus acquire at the same time both analytic and synthetic facility. To mix these two methods is distasteful to some persons. But the fact is that many theorems are more easily proved synthetically, many others analytically. In the latter case one must needs commence at the beginning with a coördinate system and work up; in the former one starts at least half way up, resting upon a large basis of known theorems. Each method has its advantages. It is still better to mingle the methods, changing from one to the other whenever the change illuminates the problem or facilitates its solution. In other words, nowadays, when a time and space limit is imposed upon us, we must at first leave aside puristic considerations of all analysis or all synthesis, and must content ourselves with developing both the analytic and synthetic methods in less time than formerly was allotted to each.

For this purpose many properties of trilinear coördinates become useless. It is of little value to know that the trilinear coördinates of a point are proportional to constant multiples of the perpendicular distances of the point from the sides of the triangle of reference. The property is not projective. On the other hand it is important to know that the ratios of the coördinates are cross ratios at the vertices or upon the sides of the triangle of reference. This property is projective and in fact assures us that the coördinate system chosen is the most general.

To shorten further and simplify the analytic presentation of projective geometry it is necessary carefully to isolate the difficulties which appear in the subject from the student's standpoint in passing from cartesian geometry. These difficulties seem to be three: The question of infinity, the fact that homogeneous instead of non-homogeneous coördinates are used, the idea and use of line coördinates in the plane or plane coördinates in space are very real difficulties. Each is a new idea to the student. They occur again and again in other geometries than projective. Beyond these three there is scarcely any other difficulty unless it be the use of abridged notation in avoiding the solution of equations. For in projective geometry as in other higher geometries one constantly aims at using expres-

sions like $u + \lambda v = 0$, $uv = 0$, determinants, etc., as much as possible to avoid tedious analysis. This can be done, although it usually is not done, in ordinary cartesian geometry.

For the past two years it has fallen to my lot to teach plane projective geometry to a class of students who know no other geometry than the euclidean and the elements of the cartesian. The allowance of time has been a half-year, forty-five lessons. This is short—perhaps too short. But when we consider the vast number of subjects, the calculus, mechanics, geometry, algebra, and perhaps physics, which demand the attention of an intermediate student of mathematics, say a junior in college, one is perhaps willing to grant that projective geometry cannot claim more. As many others may encounter similar problems in instruction the account of my experience, which comes in naturally in connection with my criticisms of Killing's Geometry, may be pardoned.

To have followed a book such as Killing's or Salmon-Fiedler's would have been to give the student not more than a bare working knowledge of trilinear coördinates of point and line. To educate his geometric insight to any great extent would have been impossible. To have followed exclusively the synthetic methods would have been to leave the student not very well grounded in them, and absolutely without the basis of analysis. It was evidently necessary, and only fair, to combine the synthetic and analytic methods.

To get the student back to where he was when he finished his Euclid, to get him again to thinking about geometric objects and geometric methods, to relieve him of the idea that geometry beyond Euclid consists merely of sets of algebraic equations of the first and second degree, it was necessary to begin with pure synthetic geometry. The subjects treated were:* Harmonic elements; the principle of duality; the elements at infinity; projective relations upon a line; involution; collineation and correlation in the plane; polarity; the conic; projective relations between ranges or pencils upon a conic. This much insures that the student thinks geometrically, that he has settled his mind about infinity and that he can solve a tolerably large number of problems by means of the fundamental theorems which he has learned.

* See my paper on "The synthetic treatment of conics" in the BULLETIN, 1903, p. 248.

The analytic treatment is then conducted as follows: Abridged notation; the introduction of homogeneous rectangular coördinates $x: y: t$, defined by $X = x: t$, $Y = y: t$; line coördinates defined as the negative reciprocals of the intercepts of a line upon the axes X and Y ; homogeneous line coördinates; the cross ratio, and its independence of projection; the fact that the ratios of the homogeneous point or line coördinates are cross ratios upon the axes or at infinity; the projection of the rectangular axes of reference and the line at infinity and the unit point into an arbitrary triangle and arbitrary point; the point and line coördinates which result; practice in the use of these coördinates. It will be seen that the difficulties of abridged notation, homogeneous coördinates and line coördinates were each treated before definitely leaving the familiar system of rectangular cartesian coördinates. It will be seen furthermore that trilinear coördinates, together with their triangle and unit point of reference, were introduced as a mere projection of customary coördinates. The two facts which allow this are the theory of collineations as explained in synthetic geometry and the invariance of the cross ratio. It is seen also that the analysis in trilinear coördinates is identical with the analysis in rectangular homogeneous coördinates, if the triangle of reference and the figure to be investigated be both projected back on the plane of X and Y . This numerical identity of the calculations in the two cases is a great help in giving the student confidence in his analysis. It is very much more valuable than the ordinary method of deducing the coördinates by the method of perpendiculars. It also avoids the necessity of the proof that the linear equation represents a straight line or point.

Generally after this method of presentation there are still a few lectures remaining for the purpose of review, to say a few words about quadratic and Cremona transformations, to take up the circular points and focal properties, or to outline the generalization to space. The student has at any rate a working knowledge of synthetic and analytic methods in the plane.

To any one who wishes to work up a course such as that outlined, the following three books: Killing's *Lehrbuch der analytischen Geometrie*, for its analysis; Enriques's *Geometria proiettiva*, for its synthetic methods; and Duporcq's *Premiers principes de géométrie moderne*, for its intermediary liveliness and its exhibition of the relations between metric and projective geometry, form a library which is quite sufficient and one

may almost say necessary. There may be other books as good ; but for this particular purpose these are not easily improved upon.

EDWIN BIDWELL WILSON.

ÉCOLE NORMALE SUPÉRIEURE,
PARIS, FRANCE,
December 5, 1902.

SHORTER NOTICES.

Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance. By M. CURTZE. Zweiter Theil. *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, XIII Heft. Leipzig, Teubner, 1902. 292 pp. 14 marks.

Abhandlungen zur Geschichte der mathematischen Wissenschaften, XIV Heft, Leipzig, Teubner, 1902. 338 pp. 16 marks.

THE first part of Curtze's *Urkunden*, forming the twelfth volume of the *Abhandlungen*, has been reviewed in the *BULLETIN* * so recently that it is quite surprising to find two new volumes of the series already published. Indeed no better evidence of the present revival of interest in the history of mathematics can be found than is seen in the encouragement recently given to this series founded a quarter of a century ago by Professor M. Cantor. The publication of the first seven volumes extended through a period of nineteen years, while the last seven, including the two under review, have appeared since 1897.

The second part of Herr Curtze's *Urkunden* is devoted to two interesting manuscripts, one the *Practica Geometriæ* of Leonardo Mainardi of Cremona, and the other the algebra of Initius Algebras. The first, which also bears the title *Leonardi Cremonensis Artis Metrice Practice Compilatio*, is a transcript, with German translation, from an Italian codex in the Venetian dialect in the university library at Göttingen. This codex is not unique, for Prince Boncompagni had two Latin manuscripts of the same work ; but not only has it never before been published, but Leonardo Mainardi has been practically unknown to historians of mathematics. It consists of fifty folios, of which the first twenty-nine and the last fourteen are here

* Vol. 9, p. 123, Nov., 1902.

transcribed and translated, the others containing merely tables of sines and a *tabula sollis* (sic), that is a table of the lengths of the various days of the year. Of the Boncompagni codices, the older belongs to the fifteenth century, although ascribed by Narducci to the fourteenth—a century before Leonardo was born. The Göttingen codex, which Herr Curtze has carefully compared with the other two, was written in 1488.

Of the life of Leonardo nothing is known save what appears in the Cremona Literata of Franciscus Arisius, published in 1702, who seems to have had access to the later of the Boncompagni codices, and who speaks of “Leonardus Maynardus Insignis Astronomus, Physicus et Mathematicus,” as having flourished in LXXXVIII, that is in 1488.

The treatise itself is not on pure geometry, but is devoted to the mensuration of heights, surfaces and solids, with such trigonometry as was necessary for the work in hand. It is divided into three “tractati,” the first being introduced with an apology for entering a field already well covered, but with the hope that this “picholo volumen” may prove a valuable compendium. This tractate is devoted to the measurement of heights, depths and lengths; the second discusses polygons and the circle, with their superficial measurement; and the third relates to the volumes of polyhedra and the three round bodies. Of the quaint mathematical terms, which of course include *umbra recta* and *umbra versa*, one of the most interesting is *columna uniforme* for a right circular cylinder or prism, a name with much to commend it. The common rules are, of course, substantially our present formulas, except for such interesting variations now and then as that for the volume of the sphere, $\frac{1}{4}d^2\pi \frac{2}{3}d$. In general the value of π is taken as $3\frac{1}{7}$, but in one case he gives the interesting approximation $3\frac{8888}{2222}$. This he obtains from the Archimedes limits by this process: $3\frac{1}{7} + \frac{1}{8}(3\frac{1}{7} - 3\frac{1}{11})$, and adds, “e questa e la proporcion del diametro a la circonferentia,” interesting as giving the older use of “proportion,” as a close approximation, as indicating that Leonardo thought this the exact value, and as inverting the terms of the ratio.

The second part of Herr Curtze’s work is entitled “Die Algebra des Initius Algebras ad Ylem geometram magistrum suum.” Of this work, almost unknown * to historians, four

* See Cantor, *Vorlesungen über Geschichte der Math.*, vol. 2, p. 589.

manuscripts are extant, no one of them the original. Of these the one in the university library at Göttingen, a manuscript of two hundred and five folios, has been followed by Herr Curtze, but it has been carefully compared with the others (all in the Dresden library) and any material differences have been noted. The book is exceedingly interesting on several accounts, mathematically and historically. Written in the south German dialect in the sixteenth century, it displays considerable knowledge together with a confusion of historic facts, and shows no little acquaintance with the algebra that was just beginning to be revived in Italy. The prologue opens: "Hic hebet sich an das Buch Algebræ,* des grossen Arismetristens, geschrieiben zu den zeithen Alexandri vnd Nectanebi, des grossen Grecken vnnd Nigromantis, geschrieiben zu Ylem, dem grossen Geometer jn Egypten, jn Arabischer Sprach genant *Gebra vnnd Almuchabola*, das dann bey vns wirdt genant *das Buch von dem Dinge der vnwissenden zall*. Vnd ist aus Arabischer Sprach jn krieichisch transferirt von Archimede, vnnd aus krieichisch jn das Latein von Apuleio, vnd wird genandt bey den Welschen *das Buch de la cosa*, das dann aber wird gesprochen *das Buch von dem ding* * * * Vnnd aus disem Buch finden wir, das der Machomet in seinem Alkoran vermeldet von disen Regeln, vnnd nennet sie auch Gebram vnd Almuchabolam. Sie werden auch gebraucht von den Indiern * * * Sie gaben auch das gemelte Buch durch etliche grunde geleutert, als es Algebras gesagt hatt, vnnd ist geschrieiben erstlichen von Algebras zu Ylem, dem grossen Geometer, der do was preceptor oder vorfarn Euclidis des fursten zu Megarien."

This passage is one of the most interesting, for its length, in the history of mathematics. The unknown Initius Algebras appears as one who lived in the fourth century B. C., contemporary with the equally unknown Ylem. The title of Alkhowarizmi's book is given, but the Arabic word *algebra*, which there first appears, is assigned to a period over a thousand years earlier and to the Greek language. Mohammed the prophet is confused with the son of Moses, ben Musa Alkhowarizmi, and the not uncommon mistake is made of taking Euclid of Megara for Euclid the Alexandrian. The early German use of "Welsch," which appears in the "Welsch practice" of the sixteenth century arithmetics, is seen; the

* I. e., the author, Initius Algebras.

uncertainty of name for the comparatively new science is apparent; the "unknown quantity" takes its place in algebra; and India is recognized as a mathematical center. Interesting as is the passage, it is no more so than many others in the work, although no other condenses so much in an equal space.

The work covers the ground of the Arab algebra, with the usual German symbolism of the sixteenth century. But the most noteworthy feature of the entire text is the reference to the solution of the cubic. That the author thought he had a solution is apparent from two of his statements,* in the second of which he expressly says that he will later show how to solve equations of the (modern) forms,

$$a + bx^2 + cx^3 = dx, \quad cx^3 + a = dx + bx^2, \quad cx^3 + dx = bx^2 + a.$$

To be sure this promise is not fulfilled, but for such a statement to be definitely made, by a writer who knew a great deal of algebra, and made before the *Ars magna* of Cardan appeared, is at least very interesting. It suggests a possibility that, after all, the Tartaglia-Cardan formula was only a rediscovery of some cherished secret of a forgotten Arab writer whose work appears in unrecognizable form in this German manuscript. Such an interpretation, at least, seems to be in Herr Curtze's mind, although to most readers it will doubtless seem more probable that the wish of the writer was father to the thought, and that it was only a dream like that of the alchemist or the circle squarer, that found place in the mind of the algebraist.

The fourteenth volume of the *Abhandlungen* contains three articles. Of these the first is by A. A. Björnbo, entitled "Studien über Melenaos' Sphärik; Beiträge zur Geschichte der Sphärik und Trigonometrie der Griechen," and served in part as the author's inaugural dissertation in 1901. It forms without question the most complete study of Menelaus that has yet been made, not only from the sources already published, but from five of the best-known Latin manuscripts of the Sphere, no Greek copy being extant, and from Arab sources as well. The author has also collated the references to Menelaus by the various Greek and Arab writers, and has critically compared the various printed editions of the work in hand, all with a scholarship that promises well for his future work in the history of mathematics.

* Pages 490, 540.

The well-known relation of the Sphere of Menelaus to the plane geometry of Euclid is brought out with a new and interesting clearness. The author concludes that the concept "spherical triangle" was introduced into mathematics by Menelaus, for although other Greek writers, including Autolycus, Euclid, Hypsicles, and Theodosius of Tripoli, had written upon the sphere, no trace of any trigonometric treatment is found in their writings. Indeed the theory of plane trigonometry may also be said to take its scientific origin in this same work, although of course the science was known before this time, notably by Hipparchus. But interesting as are all of these facts, far more so is the proof adduced to show that Menelaus recognized the projectivity of the anharmonic ratio on the sphere, and stated it as a fact known to his predecessors, probably including Hipparchus. Then with his characteristic thoroughness the author traces this proof through the Arab translators and back through the late Latin, to show how the fact happened to be lost for so many centuries.

The second part of the volume is devoted to certain "Nachträge und Berichtigungen zu 'Die Mathematiker und Astronomen der Araber und ihre Werke'" which appeared in volume 10 of the *Abhandlungen*.

The third part consists of an essay by Karl Bopp, "Antoine Arnauld, der grosse Arnauld, als Mathematiker." The idea has never been entirely wanting that Arnauld was a mathematician of power. Cantor mentions him, although only in relation to a controversy between Leibnitz and Guido Grandi; but even by his own countrymen his contributions to mathematics have long since been forgotten. Arnauld (b. 1612) lived in the time of Louis XIV, and received the name of "le grand Arnauld" because of his philosophical and theological attainments. Indeed it is in his philosophical writings that many of his mathematical ideas are suggested. It is, however, in his *Nouveaux Elémens de Géométrie* (1667) that he appears as a mathematician purely, and it is to this work that Herr Bopp devotes his chief attention. In Besoigne's *Histoire de l'abbaye de Port Royal* (1752) an anecdote of M. Nicole is given to the effect that Pascal having one day showed Arnauld a manuscript of his own on Euclid, the latter objected because the faulty arrangement of the Greek was unchanged. Thereupon Pascal challenged Arnauld to write a better work, a suggestion to which the latter acquiesced, and the "New Elements" was the result.

It is not worth while to enter into the general nature of the work. It was elementary but original. One or two of the most interesting features, however, deserve mention. The author uses in one place the expressions $\frac{1}{x_0} d'x$, $\frac{1}{y_0} d'y$, to represent these fractional parts of x and y , which leads Herr Bopp to suggest that this French symbolism might have suggested to Leibnitz his dx , an idea that is interesting even if far-fetched. He also goes more extensively into the theory of triangular numbers than had any of his predecessors, a fact which shows the influence of the Pascal-Fermat school and the overstepping of the traditional boundaries of geometry. Euclid's parallel postulate is passed with little question, for, he says, "elle a assez de clarté pour s'en contenter et ce seroit perdre de temps inutilement que de se rompre la tête pour la prouver par un long circuit," showing that his vision in this respect was no clearer than that of his contemporaries. He definitely asserts, however, the impossibility of solving the trisection problem by elementary geometry, "c'est à dire en n'y employant que des lignes droites et circulaires." Here, too, is found, five years before Pascal's publication, the latter's method of complete induction, a method which Maurolycus also understood long before either, but which was not generally appreciated. Such phrases as "it is necessary and sufficient" show that his thought was much ahead of the elementary writers of his period, while many of his proofs had a marked influence on the later French school. His generalization of propositions shed a new light on Euclid, and it is probably not too much to say that the French elementary geometry broke away from the Greek traditions largely through the influence of his initiative. The work closes with Arnauld's contributions to the theory of magic squares, a contribution that was original and powerful, and exceeded in value anything of the kind that had been attempted before his time.

DAVID EUGENE SMITH.

Annuaire pour l'an 1903, publié par le Bureau des Longitudes.
Paris, Gauthier-Villars.

THERE is, so far as we know, no scientific annual which contains anything like the amount of information which we find in the *Annuaire*. It appears to be an attempt to satisfy the needs of everyone except, perhaps, the pure mathematician. It con-

tains all kinds of astronomical information, both of temporary and permanent value for the benefit of the astronomer; information about weights and measures, and different kinds of money for the business man; tables of insurance and population, geography and statistics; all kinds of physical tables with the latest values of the physical constants; and in fact the results for everything that admits of being measured. The progress of science, however, has made the labor of the editors an increasingly difficult one. Each year of publication has seen an addition of perhaps twenty or thirty pages, until the volume has become of such a size that some new plan is necessary in order to keep it within reasonable limits.

This question has been occupying the attention of the editors for the last twelve months. It seemed undesirable to curtail the information to be inserted to any considerable extent, and at the same time it was necessary to avoid further increase. The problem has been solved on the principle of rotation. Much of the information which is given varies little, if at all, from year to year—for example, weights, measures, geography, statistics, etc.; while other portions—for example, tables of the sun and moon—must be given every year. It is thus possible so to divide the information that two successive volumes of the *Annuaire* shall give everything. In brief, every *Annuaire* will contain about 330 pages of astronomical information, part of which will be published every year, and the other part every two years. The physical constants will be given in the *Annales* of the even years; geographical and statistical constants in those of the odd years. This arrangement will come into force next year, the present number being complete in itself.

There are no special changes to note. All information is of course brought as far as possible up to date. The appendix contains the following notices: On meteorites and comets by M. Radau—an extended history of the subject up to date; a speech on the connection of science and poetry by M. Janssen; the usual report from the Mont Blanc Observatory by the same; the funeral orations of MM. Bassot and Poincaré on M. A. Cornu; and those of MM. Bassot, Bouquet de la Grye, Loewy, Janssen and Van de Sande Bakhuyzen on M. H. Faye.

ERNEST W. BROWN.

Net Premiums and Reserves on Joint Life Policies, based on the American Table of Mortality graduated by the Makeham Formula, etc., etc. By ARTHUR HUNTER. New York, The Actuarial Society of America, 1902. 4to., 102 pp.

WHILE this book is mainly filled with tables of values relating to life insurance policies based on any combination of two or three lives at risk — and is therefore designed for technical use exclusively — it may interest the readers of the BULLETIN to have pointed out to them somewhat of its scientific basis, and incidentally to note the great expansion of the life insurance business when such works of reference become necessary.

Mortality tables have been developed from crude statistics in many ways — by simple collection of data, by grading these by various summation formulæ, by arbitrary graphic interpolation and the like. Some of the tables thus obtained are incorporated in law, as the "Northampton" for valuing life interests, the "Actuaries" and the "American" for the valuation of life insurance policies. But the only scientific method which gives an algebraic expression for the law of mortality and offers great practical advantages in use has never till now been given legal recognition. This method — the Makeham Formula — gives a curve expressing the law of mortality which is continuous and adheres closely to the original data.

By former methods of graduating the crude data, the resulting tables required to complete the matter in these hundred odd pages would have filled several volumes the size of Webster's Dictionary and have been more than proportionately troublesome to use. This condensation is effected by using a curious property of Makeham's law — that of "uniform seniority" — by which instead of having to use a different table for each different combination of ages of the two lives involved in a compound risk, but one table is needed — with an auxiliary table to show the modified argument by which to enter for each varying combination of lives. In other words, under Makeham's law, for every product of the probabilities of two lives aged x and y each continuing for a year, may be substituted the square of a similar probability for one life z , z being intermediate between x and y .

Makeham's law was first established in 1860, but its practical value is just beginning to be appreciated, in this country at least.

J. M. GAINES.

NOTES.

THE thirteenth regular meeting of the Chicago section of the AMERICAN MATHEMATICAL SOCIETY will be held at Northwestern University, Evanston, Ill., on Saturday, April 11, 1903. Titles, abstracts and time requirements of papers should be in the hands of the Secretary of the Section, Professor T. F. HOLGATE, 617 Hamline St., Evanston, Ill., at least two weeks before the meeting.

THE tenth summer meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at the Massachusetts Institute of Technology, Boston, Mass., during the week beginning August 31. The last four days of the week will be devoted to a colloquium at which courses of lectures will be given as follows:

By Professor E. B. VAN VLECK, six lectures on "Selected topics in the theory of divergent series and continued fractions"; by Professor H. S. WHITE, three lectures, subject to be announced; by Professor F. S. WOODS, three lectures on "The connectivity of non-euclidean space."

A preliminary announcement giving further details concerning the summer meeting and colloquium will be issued in May.

THE Annual Register of the Society, containing a catalogue of the library, has recently been issued. Copies have been mailed to the members of the Society, and extra copies can be obtained from the Secretary.

AT the meeting of the London mathematical society held on February 12, 1903, the following papers were read: By Lieut.-Col. A. J. CUNNINGHAM: "On 4-ic residuality and reciprocity;" by Mr. E. T. DIXON: "Note on a point in the recent paper by Professor D. Hilbert;" by Mr. H. HILTON: "Some properties of binodal quartics;" by Professor A. W. CONWAY: "The field of force due to a moving electron;" by Professor W. BURNSIDE: "An arithmetical theorem connected with the roots of unity and its application to group characteristics."

THE Paris academy of sciences announces the following prizes: The Francœur prize, 1,000 francs, for a discovery or treatise advancing pure or applied mathematics.—The Poncelet prize, 1,000 francs, for that work which in the last ten years

has contributed most to the advance of pure or applied mathematics.—The Grand prize of the mathematical sciences for 1904, 3,000 francs, for perfecting in some important point the study of the convergence of continued algebraic fractions.—The Bordin prize for 1904, 3,000 francs, for developing and perfecting the theory of surfaces applicable to the paraboloid of revolution.—The Vaillant prize for 1904, 4,000 francs, for determining and studying all the displacements of an invariable figure of which the different points describe spherical curves.—The Fourneyron prize for 1903, 1,000 francs, for a theoretical or experimental study of steam turbines.—The Damoiseau prize for 1905, 2,000 francs, for the treatment of the problem: There are some ten comets whose orbits, during their visible periods, are found to be hyperbolic. To investigate, retracing the past and taking account of the perturbations of the planets, whether this was the case before the arrival of these planets in the solar system.—The Binoux prize for 1903, 2,000 francs, for works on the history of the exact sciences.—The d'Ormy prize, 10,000 francs, for pure and applied mathematics.

THE Royal academy of physical and mathematical sciences, of Naples, announces a prize of 1,000 lire for the best memoir containing a noteworthy contribution to the invariant theory of ternary biquadratic forms, preferably with reference to the various conditions for breaking up into forms of lower order. Memoirs must be submitted by June 30, 1904.

THE Hungarian academy of sciences has issued, under date of February 16, 1903, an official notice of the founding of the Bolyai prize of a medal and 10,000 crowns, to be awarded first in 1905, and every fifth year thereafter, to the author of the best work in mathematics appearing in the preceding five years.

THE annual list of prize questions of the Amsterdam scientific society has recently been issued. Copies of the announcement may be obtained from the secretary of the society, Professor D. J. KORTEWEG, Vondelstraat 104, Amsterdam.

THE several universities below offer during the summer semester of the current academic year courses in mathematics as follows:

UNIVERSITY OF GREIFSWALD. By Professor W. THOMÉ: Potential function, four hours; Theory of algebraic surfaces and space curves, two hours; Mathematical seminar, two hours.

—By Professor E. STUDY : Infinitesimal calculus, four hours ; exercises in infinitesimal calculus, one hour ; Line geometry, three hours ; Mathematical seminar.—By Professor G. KOWALEWSKI : General function theory, four hours ; Geometry of numbers, one hour ; Exercises in function theory, one hour.

UNIVERSITY OF JENA. By Professor A. GUTZMER : Differential calculus, with exercises, five hours ; Introduction to the theory of differential equations, five hours.—By Professor J. THOMAE : Analytic geometry of the plane, four hours ; Mathematical geography, four hours.—By Professor G. FREGE : Riemann's theory of functions, four hours ; Mathematical exercises, two hours.

AT the University of Nebraska, the following advanced courses in mathematics are offered during the summer session, 1903 :—By Professor E. W. DAVIS : Theory of functions.—By Professor R. E. MORITZ : History of mathematics ; Mathematical pedagogy.

ON the occasion of the Bolyai celebration at the University of Klausenburg, on January 15, the faculty conferred upon Professor H. POINCARÉ the honorary degree of doctor of philosophy.

DR. H. GRASSMANN has been promoted to a professorship of mathematics at the University of Halle.

PROFESSOR E. VOSS has been called from Würzburg to the University of Munich, as professor of mathematics.

PROFESSOR FRIEDRICH SCHOTTKY, of the University of Berlin, has been elected a member of the Berlin academy of sciences.

PROFESSOR K. HENSEL has succeeded the late Professor L. FUCHS as editor of *Crelle's Journal*.

PROFESSOR G. B. HALSTED, of the University of Texas, has been elected to the chair of mathematics at St. John's College, Annapolis, Md.

DR. M. B. PORTER, of Yale University, has been called to the professorship of mathematics in the University of Texas.

AT Columbia University, Dr. C. J. KEYSER has been promoted to an adjunct professorship of mathematics.

PROFESSOR ALEXANDER JOHNSON has resigned his position

as professor of pure mathematics in McGill University, to take effect at the close of the present academic year.

At the University of Minnesota, Dr. G. N. BAUER has been promoted to an assistant professorship of mathematics. Dr. G. A. BLISS, instructor in mathematics in the University, is absent on leave, and is spending the year at Göttingen.

PROFESSOR E. J. WILCZYNSKI, of the University of California, has received from the Carnegie Institution an appointment as research assistant and will spend a year in Europe in work on the theory of differential equations and line geometry.

MR. P. A. SMITH has resigned as instructor in mathematics at the University of Illinois to accept a position in the Hiroshima higher normal school, of Japan.

MESSRS. P. H. LINEHAN, C. W. SAXTON, and H. R. FASSETT have been appointed instructors in mathematics in the College of the City of New York.

THE Carnegie Institution has decided to provide for the publication of the works of Dr. G. W. HILL.

PROFESSOR E. L. M. CURTZE died on January 3, 1903, at Thorn, West Prussia, aged 65 years.

REAR Admiral WILLIAM HARKNESS died at New York on February 28, 1903, in his sixty-sixth year. He was for some time astronomical director of the naval observatory, and in 1897 was appointed director of the nautical almanac. In 1893 he was president of the American association for the advancement of science.

MR. W. J. C. MILLER, for many years mathematical editor of *The Educational Times* of London, died on February 11, 1903, at Bristol, England, at the age of seventy-one and a half years. During his connection with the journal mentioned, which covered a period of over thirty years, Mr. Miller published sixty-six volumes of the *Mathematical Reprint*, a series devoted chiefly to problems in modern geometry and higher algebra.

RECENT catalogues of second-hand books on mathematics: K. F. KOEHLER, 6 Kurprinzstrasse, Leipzig, Antiquariats-Katalog no. 557, 2560 titles; no. 559, 27 pages, scientific journals.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

- CARP (J. A.). Combinatorische configuraties in meerdimensionale ruimten. Utrecht, 1902. 8vo. 68 pp.
- CZUBER (E.). Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung. Leipzig, Teubner, 1903. 8vo. 15 + 594 pp. (Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. IX.) M. 24.00
- EASTON (B. S.). The constructive development of group theory; with a bibliography. Boston, Ginn, 1902. 8vo. 3 + 89 pp. Cloth. (Publications of the University of Pennsylvania, series in mathematics, No. 2.) \$0.75
- EMCH (A.). Applications of elliptic functions to problems of closure. 8vo. (*The University of Colorado Studies*, Vol. 1, pp. 81-133.)
- GANTER (H.) und RUDIO (F.). Die Elemente der analytischen Geometrie. Zum Gebrauch an höheren Lehranstalten sowie zum Selbststudium. Mit zahlreichen Uebungsbeispielen. Teil I: Die analytische Geometrie der Ebene. 5te, verbesserte Auflage. Leipzig, Teubner, 1903. 8vo. 8 + 187 pp. M. 3.00
- GRADHANDT (E.). Beiträge zur Theorie der Fokaleigenschaften der Krümmungskurven auf den Flächen zweiter Ordnung. Rostock, 1901. 8vo. 58 pp.
- HASKELL (M. W.). Die Darstellung von gewissen Resultanten in Determinantenform. 8vo. (*Jahresbericht der Deutschen Mathematiker-Vereinigung*, Band 12, pp. 38-42.)
- HEDRICK (E. R.). Supplementary note on the calculus of variations. The doctrine of infinity. 8vo. (*Bulletin of the American Mathematical Society* (2) 9, pp. 245-247 and 263-268.)
- HELWIG (P. J.). Over een algemeen gemiddelde en de integralen, die samenhangen met de foutenwet van het meetkundig gemiddelde. Utrecht, 1901. 4to. 79 pp.
- HUMBERT (G.). Cours d'analyse, professé à l'Ecole polytechnique. Vol. I. Calcul différentiel; principes du calcul intégral; applications géométriques. Paris, Gauthier-Villars, 1903. 8vo. 484 pp. Fr. 16.00
- ISSALY. La géométrie non-euclidienne et l'insuffisance de ses principes. Mémoire faisant suite au Principes fondamentaux de la théorie des pseudo-surfaces, du même auteur. Paris, Hermann, 1902. 8vo. 7 + 62 pp.
- KOIALOVICH (B. M.). On a partial differential equation of the fourth order. St. Petersburg, 1902. 8vo. 125 pp. (Russian.)

- LÜBSEN (H. B.). Ausführliches Lehrbuch der Analysis zum Selbstunterricht, mit Rücksicht auf die Zwecke des praktischen Lebens bearbeitet. 10te, verbesserte Auflage. Leipzig, Brandstetter, 1902. 8vo. 4 + 203 pp. M. 3.60
- NUGTEREN (G. K.). Rationale ruimtekrommen van de 5. ordre. Utrecht, 1901. 8vo. 67 pp.
- OVIDIO (E. D'). Geometria analitica. 3a edizione riveduta e corretta. Torino, Bocca, 1903. 8vo. 15 + 520 pp. (Biblioteca matematica, Vol. 8.) Fr. 12.00
- RUDIO (F.). See GANTER (H.).

II. ELEMENTARY MATHEMATICS.

- AGAPOV (D. V.). Collection of the most common geometrical problems with the methods of their solution. Orenburg, 1902. 8vo. 93 pp., 2 plates. (Russian.) M. 3.50
- AMALDI (U.). See ENRIQUES (F.).
- BURCH (C.). Short cuts and byways in arithmetic. Full explanations of all principles involved; examples worked out at length. Edinburgh, Blackie, 1902. 8vo. 108 pp. Cloth. 2s.
- D. (F. T.). Solutions des exercices et problèmes proposés dans le Cours d'algèbre élémentaire. (Environ 2500 questions.) Lyon and Paris, Vitte, 1902. 16mo. 452 pp.
- DIEKMANN. See KOPPE.
- ENRIQUES (F.) e AMALDI (U.). Elementi di geometria ad uso delle scuole secondarie superiori. Bologna, Zanichelli, 1902. 16mo. 22 + 655 pp. Fr. 4.50
- GREVY (A.). Eléments de géométrie, à l'usage des élèves des classes de seconde et première A et B (programmes du 31 mai 1902). Paris, Nony, 1903. 18mo. 120 pp. Fr. 1.25
- . Eléments de géométrie, à l'usage des élèves des classes de quatrième A et troisième A (programmes du 31 mai 1902). Paris, Nony, 1903. 18mo. 172 pp. Fr. 1.50
- . Géométrie à l'usage des élèves des classes de cinquième à troisième B (programmes du 31 mai 1902). Paris, Nony, 1903. 18mo. 306 pp. Fr. 2.25
- HIGGS (W. P.). Algebra self-taught; for the use of mechanics, young engineers and home students. 7th edition. New York, Spon and Chamberlin, 1903. 8vo. 104 pp. Cloth. \$0.60
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The following dates have been fixed for the meetings of the Society :

	A. M.		A. M.
Sat., Feb. 28, 1903,	11:00	Sat., Oct. 31, 1903,	11:00
" April 25, "	"	M.-Tu., Dec 28-29, "	"

The Chicago Section of the Society will meet at Northwestern University on Saturday, April 11, 1903.

The Tenth Annual Meeting and Fourth Colloquium of the Society will be held at the Massachusetts Institute of Technology, Boston, Mass., during the week beginning August 31, 1903.

BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

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OF MATHEMATICAL SCIENCE

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THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 28, 1903. The following thirty-one members of the Society were present at the morning and afternoon sessions :

Dr. Grace Andrews, Professor Joseph Bowden, Dr. J. E. Clarke, Professor F. N. Cole, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Professor Achsah M. Ely, Dr. William Findlay, Professor T. S. Fiske, Dr. A. S. Gale, Dr. E. R. Hedrick, Professor L. I. Hewes, Dr. Edward Kasner, Dr. C. J. Keyser, Dr. G. H. Ling, Mr. L. L. Locke, Dr. Emory McClintock, Professor James Maclay, Professor H. P. Manning, Professor T. F. Nichols, Professor W. F. Osgood, Miss I. M. Schottensfels, Professor D. E. Smith, Professor P. F. Smith, Mr. C. E. Stromquist, Professor H. D. Thompson, Miss Mary Underhill, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Mr. H. E. Webb, Professor R. S. Woodward.

The President of the Society, Professor Thomas S. Fiske, occupied the chair, being relieved by Vice-President Professor W. F. Osgood. The Council announced the election of the following persons to membership in the Society: Professor F. H. Bailey, Massachusetts Institute of Technology; Mr. A. T. Bell, High School, Reynolds, Ill.; Professor F. P. Brackett, Pomona College, Claremont, Cal.; Mr. W. E. Breckenridge, Morris High School, New York, N. Y.; Professor Ellen L. Burrell, Wellesley College; Miss E. B. Cowley, Vassar College; Professor E. E. De Cou, University of Oregon; Mr. F. D. Frazer, University of Oregon; Professor J. Willard Gibbs, Yale University; Dr. C. N. Haskins, Massachusetts Institute of Technology; Mr. A. C. Lunn, University of Chicago; Mr. C. L. E. Moore, Cornell University; Mr. F. G. Reynolds, College of the City of New York; Mr. C. E. Stromquist, Yale University; Professor W. E. Taylor, Syracuse University; Mr. Charles Van Orstrand, U. S. Geological Survey, Washington, D. C. Five applications for admission to the Society were received.

Professor E. W. Brown was reelected a member of the Editorial Board of the *Transactions* for a term of three years. The office of Assistant Secretary of the Society, vacated by the appointment of Dr. Edward Kasner to the editorial staff of the *Transactions*, was abolished.

The following papers were read at this meeting:

- (1) Dr. L. P. EISENHART: "Congruences of curves."
- (2) Dr. EMORY MCCLINTOCK: "The logarithm as a direct function."
- (3) Professor H. P. MANNING: "Non-euclidean geometry of nets of circles."
- (4) Mr. C. E. STROMQUIST: "A generalization of the length integral."
- (5) Dr. EDWARD KASNER: "Three notes on projective geometry."
- (6) Mr. W. B. FORD: "A theorem concerning the functions of two or more complex variables."
- (7) Professor W. F. OSGOOD: "The integral as the limit of a sum, and a theorem of Duhamel."
- (8) Dr. E. R. HEDRICK: "The integral curves of a partial differential equation."
- (9) Professor E. B. VAN VLECK: "On an extension of the 1894 memoir of Stieltjes."
- (10) Dr. A. S. GALE: "On a generalization of the set of associated minimal surfaces."
- (11) Professor G. A. MILLER: "A fundamental theorem with respect to transitive substitution groups."
- (12) Professor E. W. BROWN: "On the derivatives of the lunar coördinates with respect to the elements."
- (13) Professor CHARLOTTE A. SCOTT: "On the fundamental theorem of projective geometry."
- (14) Professor ALFRED LOEWY: "Ueber die Reducibilität der reellen Gruppen linearer homogener Substitutionen."

Professor Loewy's paper was communicated to the Society through Professor E. W. Brown. In the absence of the authors, the papers of Mr. Ford, Professor Miller, Professor Brown, Professor Scott, and Professor Loewy were read by title.

The papers of Dr. McClintock and Professor Miller will appear in the BULLETIN. Professor Scott's paper is published in the April number of the *Transactions*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. Eisenhart considers congruences of curves, the coördinates of whose points are given by the equations

$$x = f_1(t, u, v), \quad y = f_2(t, u, v), \quad z = f_3(t, u, v),$$

where u and v denote the parameters determining the curves and t is the parameter which determines points on the curve. After establishing several theorems which Darboux found from another point of view, and giving a few examples in illustration, the discussion proceeds to the problem of finding a function $\phi(u, v)$ such that the tangents to the curves of the congruence at the points of intersection with the surface defined by the above formulæ after t is replaced by ϕ , shall form a normal congruence, or in the second place, the ruled surfaces $u = \text{const.}$, $v = \text{const.}$ of this congruence of tangents shall be developable. For every congruence a function ϕ exists which furnishes a solution of the former of these problems, but not so for the second problem. However, when the curves $t = \text{const.}$, $u = \text{const.}$ on the surfaces defined by the above equations when v is constant, and when the curves $t = \text{const.}$, $v = \text{const.}$ on the surfaces $u = \text{const.}$, form a conjugate system on each, the function ϕ can be a constant, and only in this case.

When a relation between the parameters, such as $v = \phi(u)$, is given the corresponding curves form a surface and in some cases this function can be so chosen that all the tangents to these curves form a normal congruence. It is evident that the tangents to all the curves of the congruence form a complex, and the determination of congruences such that this complex be linear requires the integration of a linear differential equation of the first order.

The above expressions for x, y, z define space referred to a triple system of surfaces $t = \text{const.}$, $u = \text{const.}$, $v = \text{const.}$, and in the preceding discussion they have been considered as defining the congruence C_t of the curves of intersection of the surfaces $u = \text{const.}$, $v = \text{const.}$ It is evident however that these equations may equally well be considered as defining the congruences C_u and C_v composed of the intersections of the surfaces $t = \text{const.}$, $v = \text{const.}$ and $t = \text{const.}$, $u = \text{const.}$ respectively. On this account these equations may be said to define a *triple congruence* of curves. The theorems of the preceding discussion when viewed in this threefold light lead to new results. The existence of a triply rectilinear congruence is established and a

few of its properties found. The paper closes with a consideration of the three complexes of tangents to the curves of a triple congruence.

3. Professor Manning's paper is in abstract as follows: We can construct a geometry by taking for straight lines the circles of a net. There are three cases according as the radical center of the net is outside, on, or within all the circles. These three cases give us the hyperbolic, parabolic, and elliptic geometries, respectively. Considering only the first case, we find a real circle which cuts orthogonally all the circles of the net and which may be called the absolute. Distance between two points is proportional to the logarithm of the cross ratio of these points taken with the two points where the line joining them meets the absolute. Circles which do not meet the absolute are circles in this geometry, circles tangent to the absolute are boundary curves, and circles intersecting the absolute are straight lines or equidistant curves. Bilinear substitutions which have the absolute for invariant circle represent the three kinds of motion possible in this geometry: translation along a line and a system of equidistant curves, translation along a system of boundary curves, and rotation. Of particular interest is the group of the rotations which leave invariant a system of alternately symmetric and congruent triangles covering the plane.

4. In his dissertation (Göttingen, 1901), Georg Hamel considered those geometries in which the straight lines are the shortest distances, and determined the most general length integral for which this is the case. His work was based on the axioms laid down by Hilbert in his *Grundlagen der Geometrie*, from which he derived an expression for the length as a definite integral.

In Mr. Stromquist's paper those geometries are considered for which the circles whose centers are on the x -axis are the shortest distances. And the most general length integral (with certain restrictions which are laid down) for which this is the case is obtained in the form

$$\int_{x_1}^{x_2} \left[y^2 \int_0^{y'} \int_0^{y'} W \{ y^2 (1 + y'^2), x + yy' \} dy' dy' \right. \\ \left. + yy' \int_0^{y'} W(y^2, x) dx + \frac{d\phi(x, y)}{dx} \right] dx,$$

where W , regarded as a function of x , y , and y' ($= dy/dx$), is any continuous function satisfying the condition that $W \geq 0$ according as $-\pi/2 \leq \theta (= \tan^{-1} y') \leq +\pi/2$; and where ϕ is an arbitrary function of x and y .

This expression is then further specialized under the assumption that length $\overline{AB}$ = length $\overline{BA}$, where the length is measured along the same curve.

The condition that extremals should be perpendicular to the transversals (*i. e.*, that "radii" are perpendicular to "circles") is then discussed in general. In particular it is shown: 1° that the geometry in which straight lines are the shortest distances reduces to the euclidean geometry; 2° that the geometry in which the above circles are the shortest distances reduces to the automorphic figure (*i. e.*, the figure obtained by a conformal transformation of the pseudosphere on the plane); and 3° that, in general, any geometry in which extremals are perpendicular to the transversals may be obtained by a conformal transformation of some surface on the plane. The converse of this theorem is obviously true by Gauss's theorem.

5. Dr. Kasner's first note is entitled "A relation between projective and circular transformations." Consider a general direct circular transformation (linear transformation of the complex variable) Γ and a finite point O . The bundle of circles through O is transformed by Γ into the bundle of circles through the corresponding point O' . There is thus established a correspondence between the centers of corresponding circles. This correspondence is proved to be homographic. The projective transformation H obtained in this way depends upon both Γ and O . It is then proved that H is entirely general, *i. e.*, given any H it is possible to find a circular transformation Γ and a point O which together give rise to H . The relation between Γ , O on the one hand, and H on the other is unique, provided O is finite and not the pole of Γ , and H does not transform the line at infinity into itself. The result for the indirect circular transformations (linear transformation of the conjugate complex variable) is analogous. The relation obtained is applied to the formation of invariants of systems of circles by means of the projective invariants of their centers. Finally the relation is extended to space, but the projective transformation is no longer of the most general type.

The second note, "On the projective geometry of the plane

pentagon," deals with certain pentagons covariantly related to a given pentagon. If the vertices of the original pentagon P are a, b, c, d, e , then the *diagonal* pentagon P' is obtained by drawing the diagonals ac, bd, ce, da, eb ; and the *inscribed* pentagon P_1 is obtained by inscribing in P a conic and connecting the points of contact. The consideration of certain polarities proves Clebsch's theorem that P and P' are homographic, and also that P and P_1 are homographic. The two homographies obtained are commutative, hence the pentagon inscribed in P' coincides with the diagonal pentagon of P_1 . Repetition of the diagonal construction gives a series of pentagons $P, P', P'', \dots$ which define (at least for a convex pentagon) a definite limiting point investigated by Clebsch. The same point is obtained by indefinite repetition of the construction for the inscribed pentagon. The common limiting point may be constructed as one of the fixed points of a collineation, or as one vertex of the common self-conjugate triangle of certain conics. Corresponding points for polygons of more than five sides exist, but their theory seems to be much more complicated, if not transcendental.

The third note will appear in the BULLETIN as a separate paper.

6. In § 62 of Dini's work, *Serie di Fourier*, etc., the author establishes certain results related to the theory of functions of one complex variable, of great utility, as he subsequently shows, in the study of the convergence of infinite series. The purpose of Mr. Ford's paper is to generalize the results thus established for functions of one complex variable to the case where functions of two or more complex variables may appear. Thus a theorem is obtained which, it is believed, has important relations to the study of the convergence of a multiple series, just as Dini's results are useful in the study of similar questions pertaining to a simple series.

7. In the application of the integral calculus to problems in physics and mechanics the theorem that in the limit of a sum any infinitesimal may be replaced by another infinitesimal that differs from it by one of higher order is fundamental. The formulation which Duhamel gave for this theorem has been challenged by Mansion, who shows that it may be so understood as to lead to false results; but Mansion does not suggest a correct formulation applicable to the cases that arise in practice. The difficulties are similar to those encountered in integrating

an infinite series term by term and may be met by a requirement analogous to that of uniform convergence. The object of this paper is to give a rigorous formulation of Duhamel's theorem, and at the same time one which is adapted to the applications of the theorem which present themselves in practice.

8. Dr. Hedrick's paper forms an extension of that presented at the Christmas meeting 1901, on the characteristics of differential equations. It is found that the methods employed to obtain the characteristic lines may be developed so as to lead to a new and more rigorous treatment of integral curves. The paper is then devoted to an investigation of the cases in which the ordinary Cauchy-Kowalewski existence theorem breaks down, when applied to a general space curve.

9. The paper of Professor Van Vleck treated an extension of a memoir published by Stieltjes in two parts in the *Annales de Toulouse*, 1894-95. Stieltjes there gives an exhaustive and elegant investigation of a continued fraction

$$(1) \quad \frac{1}{a_1 z} + \frac{1}{a_2} + \frac{1}{a_3 z} + \frac{1}{a_4} + \dots$$

in which the coefficients a_n are positive. This he connects, on the one hand, with a series

$$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \frac{c_3}{z^4} + \dots$$

whose coefficients are conditioned by the relations

$$A_n \equiv \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ . & . & . & . \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix} > 0, \quad B_n \equiv \begin{vmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ . & . & . & . \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{vmatrix} > 0,$$

and, on the other hand, with one or more functional integrals

$$\Phi(z) = \int_0^\infty \frac{df(u)}{z+u}$$

in which $f(u)$ is a monotone function of u . In his extension Professor Van Vleck imposes only the condition that $a_{2n} > 0$ or that $a_{2n+1} > 0$. The corresponding restriction upon the

series is that $B_n > 0$, $A_n \neq 0$ or $A_n > 0$, $B_n \neq 0$. If the even and odd convergents are considered apart, instead of being combined by (1) into a single continued fraction, the subsidiary conditions, $A_n \neq 0$ and $B_n \neq 0$ respectively, may be dispensed with. The integrals which give rise to the continued fraction or series have the form

$$\int_{-\infty}^{\infty} \frac{df(u)}{z+u} \quad (-\infty \leq b < a \leq +\infty).$$

The condition imposed by Stieltjes upon the continued fraction is doubtless the simplest one possible. It not only suffices to ensure the reality of the roots of the numerators and denominators of the convergents, the roots lying upon the negative half of the axis of z , but it also necessitates certain theorems concerning the alternation of the roots, when two convergents are compared. These theorems, which form the basis of the investigation of Stieltjes, hold also for the more extended class of continued fractions discussed in the present paper, provided that the positive and negative half axes are considered separately. The greatest difference between the theory of Stieltjes and the extended theory appears in the consideration of questions of convergence. One new feature was a divergent series which represented in the positive and negative half planes two distinct analytic functions having the real axis as a natural boundary.

10. It is well known that a minimal surface is a surface of translation whose generators are its minimum lines. Dr. Gale considered surfaces of translation whose generators are any imaginary curves defined by functions of conjugate complex parameters. These surfaces may also be regarded as surfaces whose coördinates satisfy Laplace's equation and may therefore be legitimately spoken of as harmonic surfaces. The set of harmonic surfaces discussed is given by the equations of the set of associated minimal surfaces, omitting the conditions that the generators be minimum curves. These surfaces are intimately related to the "associated surfaces of negative curvature" which Professor Maclay discussed at the meeting of the Society on February 22, 1902. The paper will appear in the *Annals of Mathematics*.

12. The derivatives of the lunar coördinates are easily obtained when the literal expressions of the coördinates are used.

In most theories also the derivatives with respect [to all the elements except the mean motion are found without trouble. But the convergence of the series arranged in powers of the mean motions is so slow that even when we have the full numerical values and the algebraical series for a few terms the derivatives with respect to the mean motion are doubtful. Professor Brown's paper gives a method of obtaining these last derivatives accurately. Some further results are also found.

14. Professor Loewy's paper is a continuation of that by the same author published in the January (1903) number of the *Transactions*. A linear homogeneous (finite or infinite) group G with real coefficients is described as real-irreducible when no matrix P of non-vanishing determinant exists such that all the substitutions of the similar group $G' = PGP^{-1}$ have the form

$$\begin{array}{cccccccc} a_{11} & a_{12} & \cdots & a_{1m} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2m} & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} & 0 & 0 & \cdots & 0 \\ a_{m+11} & a_{m+12} & \cdots & a_{m+1m} & a_{m+1m+1} & a_{m+1m+2} & \cdots & a_{m+1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nm} & a_{nm+11} & a_{nm+12} & \cdots & a_{nn} \end{array}$$

the a 's being all real. The following theorem is established: "If a real-irreducible linear homogeneous group is not absolutely irreducible, it is similar to a decomposable group, the matrices of whose substitutions have the form

$$\begin{array}{ccccccc} c_{11} & c_{12} & \cdots & c_{1m} & 0 & 0 & \cdots & 0 \\ c_{21} & c_{22} & \cdots & c_{2m} & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{m1} & c_{m2} & \cdots & c_{mm} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & c_{11}^0 & c_{12}^0 & \cdots & c_{1m}^0 \\ 0 & 0 & \cdots & 0 & c_{21}^0 & c_{22}^0 & \cdots & c_{2m}^0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & c_{m1}^0 & c_{m2}^0 & \cdots & c_{mm}^0 \end{array}$$

the c_{ik} and c_{ik}^0 being conjugate imaginaries. Since a real-irreducible group is under consideration, all the coefficients c_{ik} in the matrices of all the substitutions cannot be real." This theorem, in combination with the results of the previous paper in the *Transactions*, serves as a basis for the study of the real-irreducible constituents of any real linear homogeneous group.

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ON THE FOUNDATIONS OF MATHEMATICS.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY AT ITS NINTH ANNUAL MEETING,
DECEMBER 29, 1902.

BY PRESIDENT ELIAKIM HASTINGS MOORE.

THE AMERICAN MATHEMATICAL SOCIETY gives its retiring President the privilege of speaking on whatever he may have at heart. Accordingly, this afternoon I propose to consider with you some matters of importance—indeed, perhaps of fundamental importance—in the development of mathematics in this country, and it will duly appear in what non-technical sense I am speaking “On the foundations of mathematics.”

I. A VIEW.

Abstract Mathematics.

The notion within a given domain of defining the objects of consideration rather by a body of properties than by particular expressions or intuitions is as old as mathematics itself. And yet the central importance of the notion appeared only during the last century—in a host of researches on special theories and on the foundations of geometry and analysis. Thus has arisen the general point of view of what may be called *abstract mathematics*. One comes in touch with the literature very conveniently by the mediation of Peano's *Revue des Mathématiques*. The Italian school of Peano and the *Formulaire Mathématique*, published in connection with the *Revue*, are devoted to the codification in Peano's symbolic language of the principal mathematical theories, and to researches on abstract mathematics. General interest in abstract mathematics was aroused by Hilbert's Gauss-Weber Festschrift, of 1899: “Ueber die Grundlagen der Geometrie,” a memoir rich in results and suggestive in methods; I refer to the reviews by Sommer,* Poincaré,† Halsted,‡ Hedrick,§ Veblen.||

* BULLETIN, vol. 6 (1900), p. 287.

† *Bull. des Sciences Mathém.*, vol. 26 (1902), p. 249.

‡ *The Open Court*, September, 1902.

§ BULLETIN, vol. 9 (1902), p. 158.

|| *The Monist*, January, 1903.

We have as a basal science logic, and as depending upon it the special deductive sciences which involve undefined symbols and whose propositions are not all capable of proof. The symbols denote either classes of elements or relations amongst elements. In any such science one may choose in various ways the system of undefined symbols and the system of undemonstrated or primitive propositions, or postulates. Every proposition follows from the postulates by a finite number of logical steps. A careful statement of the fundamental generalities is given by Padoa in a paper* before the Paris Congress of Philosophy, 1900.

Having in mind a definite system of undefined symbols and a definite system of postulates, we have first of all the notion of the compatibility of these postulates; that is, that it is impossible to prove by a finite number of logical steps the simultaneous validity of a statement and its contradictory statement. In the next place, the question of the independence of the postulates or the irreducibility of the system of the postulates; that is, that no postulate is provable from the remaining postulates. Padoa introduces the notion of the irreducibility of the system of undefined symbols. A system of undefined symbols is said to be reducible if for one of the symbols X it is possible to establish as a logical consequence of the assumption of the validity of the postulates a nominal or symbolic definition of the form $X = A$, where in the expression A there enter only the undefined symbols distinct from X . For the purpose of practical application, it seems to be desirable to modify the definition so as to call the system of undefined symbols reducible if there is a nominal definition $X = A$ of one of them X in terms of the others such that in any interpretation of the science the postulates retain their validity when instead of the initial interpretation of the symbol X there is placed the interpretation A of that symbol. If the system of symbols is reducible in the sense of the original definition it is in the sense of the new definition, but not necessarily conversely, as appears for instance from the following example, occurring in the foundations of geometry.

Hilbert uses the following undefined symbols: "point," "line," "plane," "incidence" of point and line, "incidence"

* "Essai d'une théorie algébrique des nombres entiers, précédé d'une Introduction logique à une théorie déductive quelconque." Bibliothèque du Congrès International de Philosophie, vol. 3, p. 309.

of point and plane, "between," and "congruent." Now it is possible to give for the symbol "plane" a symbolic definition in terms of the other undefined symbols — for instance, a plane is a certain class of points (as Peano showed in 1892), or again, a plane is a certain class of lines; while the notion "incidence" of point and plane receives convenient definition. It is apparent from the fact that these definitions may be given in these two ways that Hilbert's system of undefined symbols is not in Padoa's sense irreducible — at least, in so far as the symbols "plane," "incidence" of point and plane are concerned — while it is equally clear that these symbols are in the abstract geometry superfluous.

In his dissertation on euclidean geometry, Mr. Veblen, following the example of Pasch and Peano, takes as undefined symbols "point" and "between," or "point" and "segment." In terms of these two symbols alone he expresses a set of independent fundamental postulates of euclidean geometry, in the first place developing the projective geometry, and then as to congruence relating himself to the point of view of Klein in his "Erlangen Programm," whereby the group of movements of euclidean geometry enters as a certain subgroup of the group of collineations of projective geometry. Here arises an interesting question as to the sense in which the undefined symbol "congruence" is superfluous in the euclidean geometry based upon the symbols "point," "between." One sees at once that a definition of "congruence" involves parametric points in its expression, while on the other hand a definition of the system of all "planes," that is, of the general concept "plane," involves no such parametric elements. But, again, just as there exist distinct definitions of "congruence," owing to a variation of the parametric points, so there exist distinct definitions of the general concept "plane," as was indicated a moment ago. One has the feeling that the state of affairs must be as follows: In any interpretation of — say — Hilbert's symbols, wherein the postulates of Hilbert are valid, every valid statement which does not involve the symbol "plane" in direct connection with the general logical symbol ($=$) of symbolic definition, remains valid when we modify it in accordance with either of the definitions of "plane" previously referred to. On the other hand, this state of affairs does not hold for the symbol "congruence." The proof of the former statement would seem to involve fundamental logical niceties.

The compatibility and the independence of the postulates of a system of postulates of a special deductive science have been up to this time always made to depend upon the self-consistency of some other deductive science; for instance, geometry depends thus upon analysis, or analysis upon geometry. The fundamental and still unsolved problem in this direction is that of the direct proof of the compatibility of the postulates of arithmetic, or of the real number system of analysis. (To the Society this morning Dr. Huntington exhibited two sets of independent postulates for this real number system.) This is the second of the twenty-three problems listed by Hilbert in his address before the Paris Mathematical Congress of 1900.

The Italian writers on abstract mathematics for the most part make use of Peano's symbolism. One may be tempted to feel that this symbolism is not an essential part of their work. It is only right to state, however, that the symbolism is not difficult to learn, and there is testimony to the effect that the symbolism is actually of great value to the investigator in removing from attention the concrete connotations of the ordinary terms of general and mathematical language. But of course the essential difficulties are not to be obviated by the use of any symbolism however delicate.

Indeed the question arises whether the abstract mathematicians in making precise the metes and bounds of logic and the special deductive sciences are not losing sight of the evolutionary character of all life processes, whether in the individual or in the race. Certainly the logicians do not consider their science as something now fixed. All science, logic and mathematics included, is a function of the epoch — all science, in its ideals as well as in its achievements. Thus, with Hilbert, let a special deductive or mathematical science be based upon a finite number of symbols related by a finite number of compatible postulates, every proposition of the science being deducible by a finite number of logical steps from the postulates. The content of this conception is far from absolute. It involves what presuppositions as to general logic? What is a finite number? In what sense is a postulate — for example, that any two distinct points determine a line — a single postulate? What are the permissible logical steps of deduction? Would the usual syllogistic steps of formal logic suffice? Would they suffice even with the aid of the principle of mathematical induc-

tion, in which Poincaré finds* the essential synthetic element of mathematical argumentation, the basis of that generality without which there would be no science? In what sense is mathematical induction a single logical step of deduction?

One has then the feeling that the carrying out in an absolute sense of the program of the abstract mathematicians will be found impossible. At the same time, one recognizes the importance attaching to the effort to do precisely this thing. The requirement of rigor tends toward essential simplicity of procedure, as Hilbert has insisted in his Paris address, and the remark applies likewise to this question of mathematical logic and its abstract expression.

Pure and Applied Mathematics.

In the ultimate analysis for any epoch, we have general logic, the mathematical sciences,† that is, all special formally and abstractly deductive self-consistent sciences, and the natural sciences, which are inductive and informally deductive. While this classification may be satisfactory as an ideal one, it fails to recognize the fact that in mathematical research one by no means confines himself to processes which are mathematical according to this definition; and if this is true with respect to the research of professional mathematicians, how much more is it true with respect to the study, which should throughout be conducted in the spirit of research, on the part of students of mathematics in the elementary schools and colleges and universities. I refer to the articles‡ of Poincaré on the rôle of intuition and logic in mathematical research and education.

It is apparent that this ideal classification can be made by the devotee of science only when he has reached a considerable degree of scientific maturity, that perhaps it would fail to appeal to non-mathematical experts, and that it does not accord with the definitions given by practical working mathematicians. Indeed the attitude of practical mathematicians toward this

* "Sur la nature du raisonnement mathématique." *Revue de Métaphysique et de Morale*, vol. 2 (1894), pp. 371-384.

† Of which none is at present known to exist.

‡ "La logique et l'intuition dans la science mathématique et dans l'enseignement;" *L'Enseignement Mathématique*, vol. 1 (1899), pp. 157-162.

"Du rôle de l'intuition et de la logique en Mathématiques"; *Compte Rendu du Deuxième Congrès International des Mathématiciens*. (Paris [1900], 1902), pp. 115-130.

"Sur les rapports de l'Analyse pure et de la Physique mathématique"; Conference, Zurich, 1897; *Acta Mathematica*, vol. 21, p. 238.

whole subject of abstract mathematics, and especially the symbolic form of abstract mathematics, is not unlike that of the physicist toward the whole subject of theoretical mathematics, and in turn not unlike that of the engineer toward the whole subject of theoretical physics and mathematics. Furthermore, everyone understands that many of the most important advances of pure mathematics have arisen in connection with investigations originating in the domain of natural phenomena.

Practically then it would seem desirable for the interests of science in general that there should be a strong body of men thoroughly possessed of the scientific method in both its inductive and its deductive forms. We are confronted with the questions: What is science? What is the scientific method? What are the relations between the mathematical and the natural scientific processes of thought? As to these questions I refer to articles and addresses of Poincaré,* Boltzmann † and Burkhardt, ‡ and to Mach's *Science of Mechanics* and Pearson's *Grammar of Science*.

Without elaboration of metaphysical or psychological details, it is sufficient to refer to the thought that the individual as confronted with the world of phenomena in his effort to obtain control over this world is gradually forced to appreciate a knowledge of the usual co-existences and sequences of phenomena, and that science arises as the body of formulas serving to epitomize or summarize conveniently these usual co-existences and sequences. These formulas are of the nature of more or less exact descriptions of phenomena; they are not of the nature of explanations. Of all the relations entering into the formulas of science, the fundamental mathematical notions of number and measure and form were among the earliest, and pure mathematics in its ordinary acceptation may be understood to be the systematic development of the properties of these notions, in accordance with conditions prescribed by physical phenomena. Arithmetic and geometry, closely united in mensuration and trigonometry, early reached a high degree of ad-

* In addition to those already cited: "On the Foundations of Geometry"; *The Monist*, vol. 9, October, 1898, pp. 1-43. "Sur les principes de la Mécanique;" *Bibliothèque du Congrès International de Philosophie*, vol. 3, pp. 457-494.

† "Ueber die Methoden der theoretischen Physik"; Dyck's *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente*, pp. 89-98 (Munich, 1892).

‡ "Mathematisches und naturwissenschaftliches Denken"; *Jahresber. der Deutschen Math.-Ver.*, vol. 11 (1902), pp. 49-57.

vancement. But after the development of the generalizing literal notations of algebra, and largely in response to the insistent demands of mechanics, astronomy and physics, the seventeenth century, binding together arithmetic and geometry infinitely more closely, created analytic geometry and the infinitesimal calculus, those mighty methods of research whose application to all branches of the theoretical and practical physical sciences so fundamentally characterizes the civilization of to-day.

The eighteenth century was devoted to the development of the powers of these new instruments in all directions. While this development continued during the nineteenth century, the dominant note of the nineteenth century was that of critical reorganization of the foundations of pure mathematics, so far, for instance, the majestic edifice of analysis was seen to rest upon the arithmetic of positive integers alone. This reorganization and the consequent course of development of pure mathematics were independent of the question of the application of mathematics to the sister sciences. There has thus arisen a chasm between pure mathematics and applied mathematics. There have not been lacking, however, influences making toward the bridging of this chasm; one thinks especially of the whole influence of Klein in Germany and of the Ecole Polytechnique in France. As a basis of union of the pure mathematicians and the applied mathematicians Klein has throughout emphasized the importance of a clear understanding of the relations between those two parts of mathematics which are conveniently called "mathematics of precision" and "mathematics of approximation," and I refer especially to his latest work of this character, *Anwendung der Differential- und Integralrechnung auf Geometrie: Eine Revision der Principien* (Göttingen, Summer Semester, 1901; Teubner, 1902). This course of lectures is designed to present particular applications of the general notions of Klein, and furthermore it is in continuation of the discussion between Pringsheim and Klein and others, as to the desirable character of lectures on mathematics in the universities of Germany.

Elementary Mathematics.

This separation between pure mathematics and applied mathematics is grievous even in the domain of elementary mathe-

matics. In witness, in the first place: The workers in physics, chemistry and engineering need more practical mathematics, and numerous text-books, in particular, on calculus, have recently been written from the point of view of these allied subjects. I refer to the works by Nernst and Schoenflies,* Lorentz,† Perry‡ and Mellor,§ and to a book on the very elements of mathematics now in preparation by Oliver Lodge.

In the second place, I dare say you are all familiar with the surprisingly vigorous and effective agitation with respect to the teaching of elementary mathematics which is at present in progress in England, largely under the direction of John Perry, Professor of Mechanics and Mathematics of the Royal College of Science, London, and chairman of the Board of Examiners of the Board of Education in the subjects of engineering, including practical plane and solid geometry, applied mechanics, practical mathematics, in addition to more technical subjects, and in this capacity in charge of the education of some hundred thousand apprentices in English night schools. The section on Education of the British Association had its first session at the Glasgow meeting, 1901, and the session was devoted to the consideration, in connection with the section on Mathematics and Physics, of the question of the pedagogy of mathematics, and Perry opened the discussion by a paper on "The Teaching of Mathematics." A strong committee under the chairmanship of Professor Forsyth, of Cambridge, was appointed "to report upon improvements that might be effected in the teaching of mathematics, in the first instance, in the teaching of elementary mathematics, and upon such means as they think likely to effect such improvements." The paper of Perry, with the discussion of the subject at Glasgow and additions including the report of the committee as presented to the British Association at its Belfast meeting, September, 1902, are collected in a small

* Nernst und Schoenflies: *Einführung in die mathematische Behandlung der Naturwissenschaften* (Munich and Leipsic, 1895), the basis of Young and Linebarger's *Elements of Differential and Integral Calculus* (New York, 1900).

† Lorentz: *Lehrbuch der Differential- und Integralrechnung* (Leipsic, 1900).

‡ Perry: *Calculus for Engineers*. (Second edition, London, E. Arnold, 1897); German translation by Fricke (Teubner, 1902). Cf. also the citations given later on.

§ Mellor: *Higher Mathematics for Students of Chemistry and Physics*, with special reference to Practical Work. (Longmans, Green & Co., 1902. Pp. xxi + 543.)

volume, "Discussion on the Teaching of Mathematics," edited by Professor Perry (Macmillan ; second edition, 1902).*

One should consult the books of Perry, *Practical Mathematics*,† *Applied Mechanics*,‡ *Calculus for Engineers*§ and England's *Neglect of Science*,|| and his address¶ on "The Education of Engineers" — and furthermore, the files from 1899 on of the English journals, *Nature*, *School World*, *Journal of Education* and *Mathematical Gazette*.

One important purpose of the English agitation is to relieve the English secondary school teachers from the burden of a too precise examination system, imposed by the great examining bodies ; in particular, to relieve them from the need of retaining Euclid as the sole authority in geometry, at any rate with respect to the sequence of propositions. Similar efforts made in England about thirty years ago were unsuccessful. Apparently the forces operating since that time have just now broken forth into successful activity ; for the report of the British Association committee was distinctly favorable, in a conservative sense, to the idea of reform, and already noteworthy initial changes have been made in the regulations for the secondary examinations by the examination syndicates of the universities of Oxford, Cambridge and London.

The reader will find the literature of this English movement very interesting and suggestive. For instance, in a letter to *Nature* (volume 65, page 484 ; March 27, 1902), Perry mildly apologizes for having to do with the movement whose immediate results are likely to be merely slight reforms, instead of the thoroughgoing reforms called for in his pronouncements and justified by his marked success during over twenty years as a teacher of practical mathematics. He asserts that the orthodox logical sequence in mathematics is not the only possible one ; that on the contrary a more logical sequence than the orthodox one (because one more possible of comprehension by the students) is based upon the notions underlying the infinitesimal

* Cf. also "Report on the Teaching of Elementary Mathematics, issued by the Mathematical Association" (G. Bell & Sons, London, 1902).

† Published for the Board of Education by Eyre and Spottiswoode (London, 1899).

‡ D. Van Nostrand Co., New York, 1898.

§ Second edition, London, E. Arnold, 1897.

|| T. Fisher Unwin, London, 1900.

¶ In opening the discussion of the sections on Engineering and on Education at the Belfast, 1902, meeting of the British Association. Published in *Science*, Nov. 14, 1902.

calculus taken as axioms; for instance, that a map may be drawn to scale; the notions underlying the many uses of squared paper; that decimals may be dealt with as ordinary numbers. He asserts as essential that the boy should be *familiar* (by way of experiment, illustration, measurement, and by every possible means) with the ideas to which he applies his logic; and moreover that he should be thoroughly *interested* in the subject studied; and he closes with this peroration:

“ ‘Great God! I’d rather be
A pagan, suckled in a creed outworn.’ ”

I would rather be utterly ignorant of all the wonderful literature and science of the last twenty-four centuries, even of the wonderful achievements of the last fifty years, than not to have the sense that our whole system of so-called education is as degrading to literature and philosophy as it is to English boys and men.”

As a pure mathematician, I hold as the most important suggestion of the English movement the suggestion of Perry’s, just cited, that by emphasizing steadily the practical sides of mathematics, that is, arithmetic computations, mechanical drawing and graphical methods generally, in continuous relation with problems of physics and chemistry and engineering, it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry, and the calculus. This is accomplished on the one hand by the increase of attention and comprehension obtained by connecting the abstract mathematics with subjects which are naturally of interest to the boy, so that, for instance, all the results obtained by theoretic process are capable of check by laboratory process, and on the other hand by a diminution of emphasis on the systematic and formal sides of the instruction in mathematics. Undoubtedly many mathematicians will feel that this decrease of emphasis will result in much if not irreparable injury to the interests of mathematics. But I am inclined to think that the mathematician with the catholic attitude of an adherent of science in general (and at any rate with respect to the problems of the pedagogy of elementary mathematics there is no other rational attitude) will see that the boy will be learning to make practical use in his scientific investigations, to be sure in a naïve and elementary way, of the finest mathematical tools which the centuries have forged, that under

skillful guidance he will learn to be interested not merely in the achievements of the tools but in the theory of the tools themselves, and that thus he will ultimately have a feeling towards his mathematics extremely different from that which is now met with only too frequently — a feeling that mathematics is indeed itself a fundamental reality of the domain of thought, and not merely a matter of symbols and arbitrary rules and conventions.

The American Mathematical Society.

The American Mathematical Society has, naturally, interested itself chiefly in promoting the interests of research in mathematics. It has, however, recognized that those interests are closely bound up with the interests of education in mathematics. I refer in particular to the valuable work done by the committee appointed, with the authorization of the Council, by the Chicago Section of the Society, to represent mathematics in connection with Dr. Nightingale's committee of 1899 of the National Educational Association in the formulation of standard curricula for high schools and academies, and to the fact that two committees are now at work, one appointed in December, 1901, by the Chicago Section, to formulate the desirable conditions for the granting by institutions of the Mississippi Valley, of the degree of Master of Arts for work in mathematics, and the other appointed by the Society at its last summer meeting to coöperate with similar committees of the National Educational Association and of the Society for the Promotion of Engineering Education, in formulating standard definitions of requirements in mathematical subjects for admission to colleges and technological schools; and furthermore I refer to the fact that (although not formally) the Society has made a valuable contribution to the interests of secondary education in that the College Entrance Examination Board has as its Secretary the principal founder of the Society. I have accordingly felt at liberty to bring to the attention of the Society these matters of the pedagogy of elementary mathematics, and I do so with the firm conviction that it would be possible for the Society, by giving still more attention to these matters, to further most effectively the highest interests of mathematics in this country.

II. A VISION.

An Invitation.

The pure mathematicians are invited to determine how mathematics is regarded by the world at large, including their colleagues of other science departments and the students of elementary mathematics, and to ask themselves whether by modification of method and attitude they may not win for it the very high position in general esteem and appreciative interest which it assuredly deserves.

This general invitation and the preceding summary view invoke this vision of the future of elementary mathematics in this country.

The Pedagogy of Elementary Mathematics.

We survey the pedagogy of elementary mathematics in the primary schools, in the secondary schools, and in the junior colleges (the lower collegiate years). It is, however, understood that there is a movement for the enlargement of the strong secondary schools, by the addition of the two years of junior college work and by the absorption of the last two or three grades of the primary schools, into institutions more of the type of the German gymnasia and the French lycée;* in favor of this movement there are strong arguments, and among them this, that in such institutions, especially if closely related to strong colleges or universities, the mathematical reforms may the more easily be carried out.

The fundamental problem is that of *the unification of pure and applied mathematics*. If we recognize the branching implied by the very terms "pure," "applied," we have to do with a special case of *the correlation of different subjects* of the curriculum, a central problem in the domain of pedagogy from the time of Herbart on. In this case, however, the fundamental solution is to be found rather by way of indirection—by arranging the curriculum so that throughout the domain of elementary mathematics the branching be not recognized.

The Primary Schools.

Would it not be possible for the children in the grades to be trained in power of observation and experiment and reflection

* As to the mathematics of these institutions, one may consult the book on "The Teaching of Mathematics in the Higher Schools of Prussia" (New York: Longmans Green & Co., 1900) by Professor Young, and the article (BULLETIN, vol. 6, p. 225) by Professor Pierpont.

and deduction so that always their mathematics should be directly connected with matters of thoroughly concrete character? The response is immediate that this is being done to-day in the kindergartens and in the better elementary schools. I understand that serious difficulties arise with children of from nine to twelve years of age, who are no longer contented with the simple concrete methods of earlier years and who nevertheless are unable to appreciate the more abstract methods of the later years. These difficulties, some say, are to be met by allowing the mathematics to enter only implicitly in connection with the other subjects of the curriculum. But rather the material and methods of the mathematics should be enriched and vitalized. In particular, the grade teachers must make wiser use of the foundations furnished by the kindergarten. The drawing and the paper folding must lead on directly to systematic study of intuitional geometry, including the construction of models and the elements of mechanical drawing, with simple exercises in geometrical reasoning.* The geometry must be closely connected with the numerical and literal arithmetic. The cross-grooved tables of the kindergarten furnish an especially important type of connection, viz., a conventional graphical depiction of any phenomenon in which one magnitude depends upon another. These tables and the similar cross-section blackboards and paper must enter largely into all the mathematics of the grades. The children are to be taught to represent, according to the usual conventions, various familiar and interesting phenomena and to study the properties of the phenomena in the pictures; to know, for example, what concrete meaning attaches to the fact that a graph curve at a certain point is going down or is going up or is horizontal. Thus the problems of percentage, interest, etc., have their depiction in straight line or broken line graphs.

The Secondary Schools.

Pending the reform of the primary schools, the secondary schools must advance independently. In these schools at present, according to one type of arrangement, we find algebra in the first year, plane geometry in the second, physics

* Here I refer to the very suggestive paper of Benchara Branford, entitled "Measurement and Simple Surveying. An Experiment in the Teaching of Elementary Geometry" to a small class of beginners of about ten years of age (*Journal of Education*, London, the first part appearing in the number for August, 1899).

in the third, and the more difficult parts of algebra and solid geometry, with review of all the mathematics, in the fourth.

Engineers * tell us that in the schools algebra is taught in one water-tight compartment, geometry in another and physics in another, and that the student learns to appreciate (if ever) only very late the absolutely close connection between these different subjects, and then, if he credits the fraternity of teachers with knowing the closeness of this relation, he blames them most heartily for their unaccountably stupid way of teaching him. If we contrast this state of affairs with the state of affairs in the solid four years' course in Latin, I think we are forced to the conclusion that the organization of instruction in Latin is much more perfect than that of the instruction in mathematics.

The following question arises: *Would it not be possible to organize the algebra, geometry, and physics of the secondary school into a thoroughly coherent four years' course, comparable in strength and closeness of structure with the four years' course in Latin?* (Here under physics I conclude astronomy and the more mathematical and physical parts of physiography.) It would seem desirable that just as the systematic development of theoretical mathematics is deferred to a later period, likewise much of theoretical physics might well be deferred. Let the physics also be made thoroughly practical. At any rate so far as the instruction of boys is concerned, the course should certainly have its character largely determined by the conditions which would be imposed by engineers. What kind of two or three years' course in mathematics and physics would a thoroughly trained engineer give to boys in the secondary school? Let this body of material postulated by the engineer

* Why is it that one of the sanest and best-informed scientific men living, a man not himself an engineer, can charge mathematicians with killing off every engineering school on which they can lay hands? Why do engineers so strongly urge that the mathematical courses in engineering schools be given by practical engineers?

And why can a reviewer of "Some Recent Books on Mechanics" write with truth: "The student's previous training in algebra, geometry, trigonometry, analytic geometry, and calculus as it is generally taught has been necessarily quite formal. These mighty algorithms of formal mathematics must be learned so that they can be applied with readiness and precision. But with mechanics comes the application of these algorithms, and formal, do-by-rote methods, though often possible, yield no results of permanent value. How to elicit and cultivate thought is now of primary importance" ? (E. B. Wilson, *BULLETIN*, Oct., 1902). But is it conceivable that in any part of the education of the student the problem of eliciting and cultivating thought should not be of primary importance?

serve as the basis of the four years' course. Let the instruction in the course, however, be given by men who have received expert training in mathematics and physics as well as in engineering, and let the instruction be so organized that with the development of the boy in appreciation of the practical relations shall come simultaneously his development in the direction of theoretical physics and theoretical mathematics.

Perry is quite right in insisting that it is scientifically legitimate in the pedagogy of elementary mathematics to take a large body of basal principles instead of a small body, and to build the edifice upon the larger body for the earlier years, reserving for the later years the philosophic criticism of the basis itself and the reduction of the basal system.

To consider the subject of geometry in all briefness: With the understanding that proper emphasis is laid upon all the concrete sides of the subject, and that furthermore from the beginning exercises in informal deduction * are introduced increasingly frequently, when it comes to the beginning of the more formal deductive geometry why should not the students be directed each for himself to set forth a body of geometric fundamental principles, on which he would proceed to erect his geometric edifice? This method would be thoroughly practical and at the same time thoroughly scientific. The various students would have different systems of axioms, and the discussions thus arising naturally would make clearer in the minds of all precisely what are the functions of the axioms in the theory of geometry. The students would omit very many of the axioms, which to them would go without saying. The teacher would do well not to undertake to make the system of axioms thoroughly complete in the abstract sense. "Sufficient unto the day is the precision thereof." The student would very probably wish to take for granted all the ordinary properties of measurement and of motion, and would be ready at once to accept the geometrical implications of coördinate geometry. He could then be brought with extreme ease to the consideration of fundamental notions of the calculus as treated concretely and he would find those notions delightfully real

* In an article shortly to appear in the *Educational Review* on "The Psychological and the Logical in the Teaching of Geometry," Professor John Dewey, calling attention to the evolutionary character of the education of an individual, insists that there should be no abrupt transition from the introductory, intuitional geometry to the systematic, demonstrative geometry.

and powerful, whether in the domain of mathematics or of physics or of chemistry.

To be sure, as Study has well insisted, for a thorough comprehension of even the elementary parts of euclidean geometry the non-euclidean geometries are absolutely essential. But the teacher is teaching the subject for the benefit of the students, and it must be admitted that beginners in the study of demonstrative geometry cannot appreciate the very delicate considerations involved in the thoroughly abstract science. Indeed, one may conjecture that, had it not been for the brilliant success of Euclid in his effort to organize into a formally deductive system the geometric treasures of his times, the advent of the reign of science in the modern sense might not have been so long deferred. Shall we then hold that in the schools the teaching of demonstrative geometry should be reformed in such a way as to take account of all the wonderful discoveries which have been made — many even recently — in the domain of abstract geometry? And should similar reforms be made in the treatment of arithmetic and algebra? To make reforms of this kind: would it not be to repeat more gloriously the error of those followers of Euclid who fixed his *Elements* as a textbook for elementary instruction in geometry for over two thousand years? Everyone agrees that professional mathematicians should certainly take account of these great developments in the technical foundations of mathematics, and that ample provision should be made for instruction in these matters; and on reflection, everyone agrees further that this provision should be reserved for the later collegiate and university years.

The Laboratory Method.

This program of reform calls for the development of a thoroughgoing laboratory system of instruction in mathematics and physics, a principal purpose being as far as possible to develop on the part of every student the true spirit of research, and an appreciation, practical as well as theoretic, of the fundamental methods of science.

In connection with what has already been said, the general suggestions I now add will, I hope, be found of use when one enters upon the questions of detail involved in the organization of the course.

As the world of phenomena receives attention by the individual, the phenomena are described both graphically and in

terms of number and measure; the number and measure relations of the phenomena enter fundamentally into the graphical depiction, and furthermore the graphical depiction of the phenomena serves powerfully to illuminate the relations of number and measure. This is the fundamental scientific point of view. Here under the term graphical depiction I include representation by models.

To provide for the needs of laboratory instruction, there should be regularly assigned to the subject two periods, counting as one period in the curriculum.

As to the possibility of effecting this unification of mathematics and physics, in the secondary schools, objection will be made by some teachers that it is impossible to do well more than one thing at a time. This pedagogic principle of concentration is undoubtedly sound. One must, however, learn how to apply it wisely. For instance, in the physical laboratory it is undesirable to introduce experiments which teach the use of the calipers or of the vernier or of the slide rule. Instead of such uninteresting experiments of limited purpose, the students should be directed to extremely interesting problems which involve the use of these instruments and thus be lead to learn to use the instruments as a matter of course and not as a matter of difficulty. Just so the smaller elements of mathematical routine can be made to attach themselves to laboratory problems, arousing and retaining the interest of the students. Again, everything exists in its relations to other things, and in teaching the one thing the teacher must illuminate these relations.

Every result of importance should be obtained by at least two distinct methods, and every result of especial importance by two essentially distinct methods. This is possible in mathematics and the physical sciences, and thus the student is made thoroughly independent of all authority.

All results should be checked, if only qualitatively or "to the first significant figure." In setting problems in practical mathematics (arithmetical computation or geometrical construction) the teacher should indicate the amount or percentage of error permitted in the final result. If this amount or percentage is chosen conveniently in the different examples, the student will be lead to the general notion of closer and closer approximation to a perfectly definite result and thus in a practical way to the fundamental notions of the theory of limits and of ir-

rational numbers. Thus, for instance, uniformity of convergence can be taught beautifully in connection with the concrete notion of area under a monotonic curve between two ordinates, by a figure due to Newton, while the interest will be still greater if in the diagram area stands for work done by an engine.

The teacher should lead up to an important theorem gradually in such a way that the precise meeting of the statement in question, and further, the practical — *i. e.*, computational or graphical or experimental — truth of the theorem is fully appreciated; and furthermore, the importance of the theorem is understood, and indeed the desire for the formal proof of the proposition is awakened, before the formal proof itself is developed. Indeed, in most cases, much of the proof should be secured by the research work of the students themselves.

Some hold that absolutely individual instruction is the ideal, and a laboratory method has sometimes been used for the purpose of attaining this ideal. The laboratory method has as one of its elements of great value the flexibility which permits students to be handled as individuals or in groups. The instructor utilizes all the experience and insight of the whole body of students. He arranges it so that the students consider that they are studying the subject itself, and not the words, either printed or oral, of any authority on the subject. And in this study they should be in the closest coöperation with one another, and with their instructor, who is in a desirable sense one of them and their leader. Instructors may fear that the brighter students will suffer if encouraged to spend time in coöperation with those not so bright. But experience shows that just as every teacher learns by teaching, so even the the brightest students will find themselves much the gainers for this coöperation with their colleagues.

In agreement with Perry, it would seem possible that the student might be brought into vital relation with the fundamental elements of trigonometry, analytic geometry, and the calculus, on condition that the whole treatment in its origin is and in its development remains closely associated with thoroughly concrete phenomena. With the momentum of such practical education in the methods of research in the secondary school, the college students would be ready to proceed rapidly and deeply in any direction in which their personal interests might lead them. In particular, for instance, one might ex-

pect to find effective interest on the part of college students in the most formal abstract mathematics.

For all students who are intending to take a full secondary school course, in preparation for colleges or technological schools, I am convinced that the laboratory method of instruction in mathematics and physics, which has been briefly suggested, is the best method of instruction, for students in general, and for students expecting to specialize in pure mathematics, in pure physics, in mathematical physics or astronomy, or in any branch of engineering.

Evolution, not Revolution.

In contemplating this reform of secondary school instruction we must be careful to remember that it is to be accomplished as an evolution from the present system and not as a revolution of that system. Even under the present organization of the curriculum the teachers will find that much improvement can be made by closer coöperation one with another; by the introduction so far as possible of the laboratory two-period plan; and in any event by the introduction of laboratory methods; laboratory record books, cross section paper, computational and graphical methods in general, including the use of colored inks and chalks; the coöperation of students; and by laying emphasis upon the comprehension of propositions rather than upon the exhibition of comprehension.

The Junior Colleges.

Just as the secondary schools should begin to reform without waiting for the improvement of the primary schools, so the elementary collegiate courses should be modified at once without waiting for the reform of the secondary schools. And naturally, in the initial period of reform, the education in each higher domain will involve many elements which later on will be transferred to the later domain.

Further, by the introduction into the junior colleges of the laboratory method of instruction it will be possible for the colleges and universities to take up a duty which for the most part has been neglected in this country. For, although we have normal schools and other training schools for those who expect to teach in the grades, little attention has as yet been given to the training of those who will become secondary

school teachers. The better secondary schools to-day are securing the services of college graduates who have devoted special attention to the subjects which they intend to teach, and as time goes on the positions in these schools will as a rule be filled (as in France and Germany) by those who have supplemented their college course by several years of university work. Here these college and university graduates proceed at once to their work in the secondary schools. Now in the laboratory courses of the junior college let those students of the senior college and graduate school who are to go into the teaching career be given training in the pedagogy of mathematics according to the laboratory system ; for such a student the laboratory would be a laboratory in the pedagogy of mathematics ; that is, he would be a colleague-assistant of the instructor. By this arrangement, the laboratory instruction of the colleges would be strengthened at the same time that well equipped teachers would be prepared for work in the secondary schools.

The Freedom of the Secondary Schools.

The secondary schools are everywhere preparing students for colleges and technological schools, and whether the requirements of those institutions are expressed by way of examination of students or by way of the conditions for the accrediting of schools or teachers, the requirements must be met by the secondary schools. The stronger secondary school teachers too often find themselves shackled by the specific requirements imposed by local or by collegiate authorities. Teaching must become more of a profession. And this implies not only that the teacher must be better trained for his career, but that also in his career he be given with greater freedom greater responsibility. To this end closer relations should be established between the teachers of the colleges and those of the secondary schools ; standing provision should be made for conferences as to improvement of the secondary school curricula and in the collegiate admission requirements ; and the leading secondary school teachers should be steadily encouraged to devise and try out plans looking in any way toward improvement.

Thus the proposed four years' laboratory course in mathematics and physics will come to existence by way of evolution. In a large secondary school, the strongest teachers, finding the

project desirable and feasible, will establish such a course alongside the present series of disconnected courses — and as time goes on their success will in the first place stimulate their colleagues to radical improvements of method under the present organization and finally to a complete reorganization of the courses in mathematics and physics.

The American Mathematical Society.

Do you not feel with me that the AMERICAN MATHEMATICAL SOCIETY, as the organic representative of the highest interests of mathematics in this country, should be directly related with the movement of reform? And, to this end, that the Society, enlarging its membership by the introduction of a large body of the strongest teachers of mathematics in the secondary schools should give continuous attention to the question of improvement of education in mathematics, in institutions of all grades? That there is need for the careful consideration of such questions by the united body of experts, there is no doubt whatever, whether or not the general suggestions which we have been considering this afternoon turn out to be desirable and practicable. In case the question of pedagogy does come to be an active one, the Society might readily hold its meetings in two divisions — a division of research and a division of pedagogy.

Furthermore, there is evident need of a national organization having its center of gravity in the whole body of science instructors in the secondary schools; and those of us interested in these questions will naturally relate ourselves also to this organization. It is possible that the newly-formed Central Association of Physics Teachers may be the nucleus of such an organization.

III. CONCLUSION.

The successful execution of the reforms proposed would seem to be of fundamental importance to the development of mathematics in this country. I urge that individuals and organizations proceed to the consideration of the general question of reform with all the related questions of detail. Undoubtedly in many parts of the country improvements in organization and methods of instruction mathematics have been making these last years. All persons who are, or may become, actively interested in this movement of reform should in some way

unite themselves in order that the plans and the experience, whether of success or of failure, of one may be immediately made available in the guidance of his colleagues.

I may refer to the centers of activity with which I am acquainted. Miss Edith Long, in charge of the Department of Mathematics in the Lincoln (Neb.) High School, reports upon the experience of several years in the correlation of algebra, geometry, and physics, in the October (1902) number of the *Educational Review*. In the Lewis Institute of Chicago, Professor P. B. Woodworth, of the Department of Electrical Engineering, has organized courses in engineering principles and electrical engineering in which are developed the fundamentals of practical mathematics. The general question came up at the first meeting* (Chicago, November, 1902) of the Central Association of Physics Teachers, and it is to be expected that this association will enlarge its functions in such a way as to include teachers of mathematics and of all the sciences, and that the question will be considered in its various bearings by the enlarged association. At this meeting informal reports were made from the Bradley Polytechnic Institute of Peoria, the Armour Institute of Technology of Chicago, and the University of Chicago. The question is evoking much interest in the neighborhood of Chicago.

I might explain how I came to be attracted to this question of pedagogy of elementary mathematics. I wish, however, merely to express my gratitude to many mathematical and scientific friends, in particular, to my Chicago colleagues, Mr. A. C. Lunn and Professor C. R. Mann, for their coöperation with me in the consideration of these matters, and further to express the hope that we may secure the active coöperation of many colleagues in the domains of science and of administration, so that the first carefully chosen steps of a really important advance movement may be taken in the near future.

I close by repeating the questions which have been engaging our attention this afternoon. In the development of the individual in his relations to the world, there is no initial

* Subsequent to the meeting of organization in the spring of 1902. Mr. Chas. H. Smith, of the Hyde Park High School, Chicago, is president of the Association. Reports of the meetings are given in *School Science* (Ravenwood, Chicago).

separation of science into constituent parts, while there is ultimately a branching into the many distinct sciences. The troublesome problem of the closer relation of pure mathematics to its applications: can it not be solved by indirection, in that through the whole course of elementary mathematics, including the introduction to the calculus, there be recognized in the organization of the curriculum no distinction between the various branches of pure mathematics and likewise no distinction between pure mathematics and its principal applications? Further, from the standpoint of pure mathematics: will not the twentieth century find it possible to give to young students during their impressionable years in thoroughly concrete and captivating form the wonderful new notions of the seventeenth century? By way of suggestion these questions have been answered in the affirmative, on condition that there be established a thoroughgoing laboratory system of instruction in primary schools, secondary schools, and junior colleges — a laboratory system involving a synthesis and development of the best pedagogic methods at present in use in mathematics and the physical sciences.

CONCERNING THE AXIOM OF INFINITY AND MATHEMATICAL INDUCTION.

BY PROFESSOR C. J. KEYSER.

(Read before the American Mathematical Society, December 29, 1902.)

I. *Introductory Considerations.*

This paper deals with a question which, on the one hand, is a question of pure logic, and, on the other, a question of *Mengenlehre*. It is often asserted, and is probably true, that reasoning naturally takes place in accordance with what the logicians of the school called first intentions. But ratiocination as activity, however unconscious its conformation to law, is nevertheless not lawless; and from the period when this fact came clearly into the consciousness of the Greek mind, as early as the time of Protagoras,* science has been neither able nor

* The so-called laws of thought seem to have struggled into consciousness mainly through the disputations of the Sophists. The law of contradiction, in particular, appears to have received its earliest formulation in the *καταβάλλοντες* of Protagoras. Cf. Windelband: *Geschichte der Philosophie*, and Ueberweg: *System der Logik* (both works also in English).

willing to escape the consideration of second intentions in their logical significance. So true is this that, despite the age-long tyranny exercised by the Aristotelian logic—a tyranny having, at least in the domain of science, scarcely a match except in the case of Euclid's elements—the forms of thought, which serve as a kind of diagrammatic representation of the orderliness of the reasoning processes, sustain to-day perhaps even greater interest than ever before. The mathematician's interest in these forms is two-fold, attaching to them both as norms for mechanically testing the validity of arguments and as constituting exceedingly subtle matter for mathematical investigation.

Of all the argument-forms, there is one which, viewed as the figure of the way in which the mind gains certainty that a specified property belongs to each element of a given assemblage, enjoys the distinction of being at once perhaps the most fascinating, and, in its mathematical bearings, doubtless the most important, single form in modern logic. I refer to the argument-form variously known as reasoning by recurrence, induction by connection (De Morgan), mathematical induction, complete induction, and Fermatian* induction—a form of procedure unknown to the Aristotelean system, for this latter allows apodictic certainty in case of deduction only, while it is just the characteristic of complete induction that it yields such certainty by the reverse process, a movement from the particular to the general, from the finite to the infinite.

That the highest degree of certainty is thus attainable has been the living faith of mathematicians at least since the time of Fermat, and on it is based the whole modern movement towards the rigorization or, at all events, the arithmetization of mathematics, for, as is well known, it is precisely by means of complete induction that the fundamental laws of number have been or admit of being established for the totality of integers. It is accordingly not a matter for surprise that logicians, whether of the traditional or of the modern so-called "exact" † school, have felt challenged to examine the method in question and to seek an adequate *a priori* justification for the mentioned faith in its validity.

* So named by Mr. C. S. Peirce, according to whom this form of reasoning is due to Fermat.

† A minute analysis of the method in question, in connection with the presuppositions of Dedekind's proof of the "theorem of complete induction," is found in Schröder's *Algebra der Logik*, vol. 3.

Among the discussions of the subject that are readily accessible to mathematicians not familiar with the symbology of technically mathematical logic, the most notable are that by Dedekind in his *Was sind und Was sollen die Zahlen* (also in English, 1901) and that by Poincaré in his article "Sur la nature du raisonnement mathématique" † (*Revue de Métaphysique et de Morale*, volume II). The latter essay, though written (1894) several years after the publication of the former, makes no allusion to it. The two discussions have in fact little in common save their problem, which is that of disclosing what the German calls "die wissenschaftliche Grundlage" of a mode of logical procedure characterized by the Frenchman as "le raisonnement mathématique par excellence." They approach their common task from the most widely sundered points of departure and along paths which conduct them finally to views that, as will subsequently appear, are neither coincident nor, strictly regarded, compatible. It is this striking difference of method and especially the essential though elusive difference of conclusion, which we plead as an excuse, or, at all events, as a sufficient provocation for undertaking to examine the matter once more. Such an examination will naturally involve a critical review and comparison of the discussions in question.

II. Poincaré's View.—The Axiom of Infinity.

For Poincaré the question involved is of the farthest-reaching critical importance. To answer it is nothing less than to show not merely the logical validity but the logical possibility of mathematical science, because for such possibility it is necessary to be able to establish *general* theorems, *i. e.*, to gain the the highest degree of certainty that a specified property belongs to each element of an infinite assemblage, an achievement to which the analytic, or syllogistic, method, dependent as it is upon the axioms of identity, contradiction and excluded third, is inadequate. For while these axioms validate deduction, inference from the general to the particular, it is precisely these, when regarded as the *sole* axioms of formal thought, which invalidate the inverse process. If this inverse process be not logically performable, then either mathematics neither is, nor

† An interesting discussion of Poincaré's paper by G. Léchâles is found in the same volume of the same *Revue*.

can become, a logically rigorous science or else it reduces in last analysis "à une immense tautologie."

Now the mathematician affirms the performability of the process in question, namely, in accordance with the argument-form, complete induction. What, then, is, according to Poincaré, the logical ground of this affirmation?

We are first told what that ground is not. Suppose it established, in regard to some property p : (1) that p belongs to the number 1; (2) that if p belongs to an integer n , it belongs to $n + 1$. Propositions (1) and (2) afford the means of generating one after another a sequence of syllogisms by which one proves first that p belongs to 2, then to 3, and so on. In order to ascertain by this analytic method whether p belongs to a specified integer m , it is necessary to determine in advance the same question for each of the integers 2, 3, $\dots$, $m - 1$, in the order as written, a process requiring a number of syllogisms which is greater the greater the number m . Accordingly, *this* method, of successive deductions, is not available for determining whether p is a property of each in the *totality* of integers. Equally powerless to that end is experience (including observation) for this can take account of the individuals of a finite assemblage at most. Either analysis or experience may succeed if a sequence be finite but if it be infinite both must fail. Not less vain is it to invoke finally the aid of induction as employed in the physical sciences, for this latter, resting upon a purely assumed order in the external universe, is confessedly *inductio imperfecta* and as such can yield approximate certainty only.

Nevertheless, despite the inadequacy of the means mentioned, as soon as hypotheses (1) and (2) are granted and the indicated sequence of deductions is *begun*, "the judgment imposes itself upon us with irresistible evidence" that p is a property of *all* the integers. Why? It appears to be clear that the answer must be the adduction of an additional presupposition of formal thought, a presupposition whose *formulation* shall mark a *conscious* extension of the domain of logic by affirming as axiomatic that apodictic certainty can transcend every limited sequence of deductions or observations. Such presupposition, which I venture to call the axiom of infinity, is stated by Poincaré, in answer to the foregoing question "why," as follows: "*C'est qu'il n'est que l'affirmation de la puissance de l'esprit qui se sait capable de concevoir la répétition indéfinie d'un même acte dès que cet acte est une fois possible.*"

Is the axiom sufficient? We are not explicitly told precisely what the operation (acte) is which in the present case it is at once possible and necessary to conceive as indefinitely repeated. To examine the matter, observe that, with a view to availing ourselves below of the familiar simplicity of the first Aristotelian figure, proposition (2) may be stated categorically as follows: every integer next after an integer having the property p is an integer having the property p . Now denote by P the expression: an integer having the property p ; by N the expression: integer next after P ; and construct the pair of syllogisms

$$(\alpha) \left\{ \begin{array}{l} \text{(i) } k \text{ is } P, \\ \text{(ii) } k + 1 \text{ is an integer next after } k, \\ \therefore k + 1 \text{ is an } N; \end{array} \right. \left\| \begin{array}{l} \text{(iii) every } N \text{ is } P, \\ \text{(iv) } k + 1 \text{ is an } N, \\ \therefore k + 1 \text{ is } P. \end{array} \right.$$

(A) These syllogisms, being formally valid, will be valid no matter what the meaning of k , and will, therefore, be valid if k be replaced by $k + 1$. Such substitution yields a pair (α') of syllogisms which may be described as the pair next after the pair (α). *The first premiss of (α') is the last conclusion of (α).*

Let k denote an integer, then (ii) is true by virtue of the assumed sequence of integers, and (iii) being true by (2), it is seen that the last conclusion of (α) is true provided the first premiss is true. By help of (A) it follows that the last conclusion of (α') is true if the first premiss (i) of (α) is true, and the same holds for the last conclusion of (α''), pair next after (α'), if such pair (α'') be supposed constructed, and so on; i. e., if (i), first premiss of (α), be true, then will be true the last conclusions of all pairs of any sequence (α), (α'), (α''), $\dots$, either actually or conceptually constructed. Now if $k = 1$, then, by (1), (i) is true. Accordingly to gain certainty that every integer is P , it is sufficient to construct one after another a syllogistic pair for every integer, which is impossible, or to conceive it done, which is possible by axiom. Thus it is seen that the operation to which the axiom of infinity is to be applied in the present case is the operation O (beginning with (α) for $k = 1$) where O denotes the construction of a syllogistic pair next after the pair last constructed. This conclusion appears to differ from that of Poincaré* in that he appears to hold that

* Cf. *op. cit.*, p. 379, V.

the operation in question is the construction of one syllogism instead of a pair. At all events, the operation is, in form, a *deduction*, and, for the following criticism, it is quite indifferent whether it be one or a pair.

The operation O is obviously composite, and for the present purpose may be resolved into two operations O_1 and O_2 , where O_1 (beginning with 1) means assigning to k the integer next after the integer last assigned, and O_2 (beginning with 1) means substituting for k in (α) the (by O_1) assigned integer next after the integer last substituted in (α) . Now of these component operations, the former is not deductive (*i. e.*, not “analytique,” not syllogistic), while the latter is deductive, being a process of constructing arguments of type (α) . The operation O_2 presupposes O_1 . The question arises: *Is it necessary to apply the axiom of infinity to both O_1 and O_2 ?*—a question to be answered in the following section.

III. Examination of Dedekind's View.

Let S be a system of elements such that there is a law ϕ of depiction (Abbildung) depicting S upon itself so that each element e of S is depicted upon one and but one element e' of S and that no two elements are depicted upon a same element. Call e' the picture (Bild) or image of e . Every part of S (including S itself as a special case) thus depicted upon itself is named chain (Kette) under ϕ . Denote by A an arbitrary part of S and by A_0 the assemblage of the elements common to all chains (in S) containing A . It is obvious that, S and ϕ being granted, A_0 exists for every A , and Dedekind proves that A_0 is itself a chain, and describes it as the chain of A under ϕ . Let Σ be an assemblage.

THEOREM.—*In order to prove that A_0 is part of Σ it is sufficient to prove: (σ) that A is part of Σ , and (ρ) that the image of every element of A_0 belonging to Σ belongs to Σ .*

An apparently simpler proof than that by Dedekind runs thus: Let $A_0 \equiv A_1 + A_2$, where A_1 denotes the assemblage of all those elements of A_0 that belong to Σ . By (ρ) , A_1 is a chain, and, by (σ) , contains A . Hence, by definition of A_0 , A_2 has no element, and $A_0 \equiv A_1$.

Such is the beautiful theorem which the author characterizes — with what justification, we shall seek to determine at a later stage — as “die wissenschaftliche Grundlage” of complete induction. It affords the means of answering the closing ques-

tion of II. If we denote by S the assemblage of all integers, by A the number 1, and by Σ the assemblage of all things each having the property p , then, the foregoing hypotheses (1) and (2) being granted, the proposition p belongs to *every* integer, by Dedekind's theorem follows immediately on finding a ϕ (depicting n on $n + 1$) under which $S \equiv A_0$ of 1. Now precisely this identity is established by applying the axiom of infinity to the above defined operation O_1 . Accordingly Poincaré's *application of the axiom to the analytic operation O_2 is tautological*. Indeed it is superfluous to use O_2 even one time, and it appears that for Poincaré's problem the axiom might as well be reworded so as to restrict its application to O_1 . So restricted, it is necessary, as seen, for Dedekind in the same connection. The supposed restriction, however, regarded as excluding O_2 , is rather apparent than genuine, for if one initially assumes, and this is possible without affecting the sufficiency of the application to O_1 , that the k of O_1 is in the diagrammatic scheme (α), then the application to O_1 carries with it the application to O_2 . Next note that such deductive forms as (α) are presupposed by Dedekind, being in fact employed by him in proving his theorem. Accordingly, Dedekind's proof of the proposition, p belongs to every integer, is *not more fundamental* than the application of Poincaré's axiom to O_2 , and hence not more fundamental than Poincaré's proof of the same proposition; than Poincaré's "proof," I say, for while the application of his axiom to O_2 is not necessary *with*, it is, we have seen, sufficient *without*, Dedekind's theorem—a fact which will presently be seen to be decisive against the latter author's claim of logical priority for his theorem. For we can now show that this theorem, so far from being "*the scientific basis of*," admits of being proved by, the method of mathematical induction. To this end we establish the

THEOREM (a).—*Every chain A_0 of a part A of a system S under a ϕ consists of a denumerable assemblage of assemblages*

$$A, A_1, A_2, \dots, A_{n-1}, A_n, A_{n+1}, \dots$$

where A_n is the assemblage of those images of the elements of A_{n-1} that are not in A .

For denote by A_1 the assemblage of elements that serve as images of all such elements of A as are not imaged on elements of A . A_0 , since it contains A and is a chain, contains A_1 . Denote by A_2 the assemblage of the elements that serve as

images of those elements of A_1 that are not imaged on elements of A . It is plain that A_2 is in A_0 , and, by definition of ϕ , that no element is common to A_1 and A_2 . The sequence of A 's thus generable (or rather thus brought to attention one after another, for the A 's are already generated by ϕ) may or may not have an end. In either case the assemblage E of all the elements in the A 's obviously is a chain (in S) under ϕ . The A 's being in A_0 , so is E ; and A being in chain E , so, by its definition, is A_0 ; $\therefore E \equiv A_0$.

Now assume Dedekind's data (σ) and (ρ) . From (ρ) follows (ρ') : if A_n belongs to Σ , then so does A_{n+1} . If now we regard (σ) and (ρ') as our data, then it follows, by Poincaré's axiom, that every one of the A 's and therewith $E(\equiv A_0)$ belongs to Σ . And so, it appears, the so-called foundation of ordinary mathematical induction is susceptible of being laid by ordinary mathematical induction.

In passing it may be noted that in the sequence of theorem (a) the power (in Cantor's sense) of none of the A 's can be higher, though in each of those following some one it may be lower, than that of A . Accordingly, if we assume the proposition that the power of the assemblage of all the elements of the assemblages of a denumerable assemblage of assemblages having each of them a same power α , is α , then follows the

THEOREM (b).—*The power of the chain A_0 of any given part A of any given assemblage S under a given ϕ , is the same as the power of A .**

It is proper here to recognize a fact emphasized by Schröder † that Dedekind establishes his theorem without making use of either the notion of "number" or the notion of the number "series." On the other hand, the foregoing proof of the theorem by ordinary mathematical induction employs both these notions. But this difference does not justify the claim of relative fundamentality for that theorem, for the two notions mentioned, although they are by Dedekind introduced after, are not introduced through, the theorem, their being as concepts depending upon it neither mediately nor immediately.

Dedekind's theorem viewed as a generalization. Let (γ) stand for the conclusion: A_0 belongs to Σ . Then the theorem, T , is: if (σ) and (ρ) are known, (γ) can be inferred. In form this is

* The fact that A may be finite and A_0 denumerably infinite is in spirit hardly an exception.

† Cf. *op. cit.*, p. 355.

identical with that of mathematical induction as ordinarily understood. But the latter, call it I , deals with only a denumerable assemblage, ϕ depicting n on the next, $n + 1$; while A_0 , subject of T , may have any power whatever (though Dedekind seems to think of no power higher than that of the continuum) and the image e' of e may not be next to e . Accordingly T , dispensing with the notion of nextness essential to I , is a generalization of I . Now the T data, (σ) and (ρ) , are sufficient for I , for as we have seen, the I data, (σ) and (ρ') , are consequences of (σ) and (ρ) . Accordingly, although I is therefore available whenever T is available, T has nevertheless, especially in case A_0 is non-denumerable, a certain advantage over I : (γ) is yielded immediately by T but only mediately by I (cf. theorem (a)). On the other hand since (σ) and (ρ) are not consequences of (σ) and (ρ') , I may be available when T is not. So it appears that T , regarded as in the sense explained a generalization of I , is, as an engine of investigation, inferior to I .

The last conclusion hinges on the yet improved statement that (ρ) is not a consequence of (ρ') . It will be sufficient to verify the statement by a simple example. Consider the assemblage D of the elements d of assemblages D_k ($k = 1, 2, \dots$). Suppose D depicted upon itself by a ϕ so that every d of D_n is an image d' of one and but one d of D_{n-1} and that every d of D_{n-1} is imaged in either D_n or D_1 , but not in both. Under ϕ , D is obviously D_0 of D_1 . For clearness we may suppose the d 's to be delegates to a nominating convention. Now conceivably it may be known at once: (σ) that every d of D_1 will vote for C ; (ρ') that every d of D_n will vote for C if every d of D_{n-1} will do so; and that if some d will vote, not for C , either it is unknown how the d' of that d will vote or that d' will vote, not for C ; i. e., under ϕ the I data (σ) and (ρ') are given while the T hypothesis (ρ) is either known to be false or not known to be true. Accordingly, if Σ be the assemblage of d 's who will vote for C , I avails while T fails to prove that D belongs to Σ .

IV. *Circularity of the Bolzano and Dedekind Proofs of the Existence* Theorem for Infinite Assemblages.*

Bolzano's definition of infinite assemblage, introduced by aid of various subtle preliminaries (§§ 3-9),† amounts to this: an

* For citation and analysis of the modern mathematical literature concerned with the question of actual infinity, cf. Veronese: *Grundzüge der Geometrie*. Note IV.

† Bolzano: *Paradoxien des Unendlichen*.

assemblage is infinite if, and only if, it cannot be exhausted by removing from it, one after another, finite (§ 8) assemblages of its elements. In § 13, a proof is attempted of the proposition that such an assemblage exists, namely *die Menge der Sätze und Wahrheiten an sich*. The attempt informally postulates: the proposition, such truths exist, is such a truth, A ; A is true, is another such truth, B ; so on; and, the indicated process is inexhaustible. The last, an evident *petitio principii*, is doubtless to be granted, but only under some such axiom as that of Poincaré.

Bolzano affirms and exemplifies (§ 20), though he does not demonstrate, the proposition that every infinite assemblage can be paired in one-one fashion with a proper part of itself—a property employed independently by Dedekind as the defining property of the infinite and yielding a definition shown by Dedekind and independently by others* to be equivalent to the foregoing one of Bolzano's. By virtue of the "intrinsic" character of Dedekind's definition, his proof of the existence theorem better conceals, though it undoubtedly contains, a logical *petitio*. On examination the proof† is seen to postulate as certainties: (1) if there be a t ,‡ there is a t' (call it image of t) having t as object; (2) if there be two distinct t 's, the corresponding t 's are distinct; (3) there is a t ; (4) there is a t which is not a t' ; (5) every t is another t than its t' . These being granted, it hardly follows deductively, though it is tacitly and, we may allow, admissibly § assumed, that there is an assemblage ϑ of t 's, a totality excluding none of them. There is, then, plainly a ϕ depicting each t on its t' and therewith, by virtue of (4), depicting ϑ on a proper part of itself; $\therefore \vartheta$ is infinite. Now, provided one be permitted to reflect, it equally follows, from the postulates, that ϑ contains a sequence S of t 's beginning with the t of (4), each succeeding t being the t' of its predecessor. S , too, is infinite by Dedekind's definition. Now certainty (1), in its character as a certainty postulated a priori, can not be contingent upon conclusions (such as that regarding ϑ or S) to be subsequently drawn from it joined to like certainties. Hence, even if the other postulates be rejected, certainty

* Cf. "Concerning positive definitions of finite assemblage, and infinite assemblage," BULLETIN, vol. VII.

† Cf. *op. cit.*, § 66.

‡ The symbol t standing for the word thought.

§ Such postulates as (1) and (2) seem to depend for their intelligibility upon the notion of totality.

(1) involves certainty that the imaging process shall not fail even though as yet perhaps unknown considerations may demand that it be endlessly performable. Accordingly (1) involves a statement included in Poincaré's axiom, which appears indeed to be a presupposition of all logical discourse, the existence of the infinite being unavoidably, however unconsciously, assumed, and so not demonstrable.

COLUMBIA UNIVERSITY.

A GERMAN CALCULUS FOR ENGINEERS.

Hauptsätze der Differential- und Integral-Rechnung. Von DR. ROBERT FRICKE. Dritte umgearbeitete Auflage. Braunschweig, Vieweg, 1902. 4vo., 218 pp.

WHILE the needs of American technical schools, and their environment, render a foreign book on the calculus unsuitable for use as a text, the difficult questions which arise in regard to the methods of presentation of this subject are largely the same throughout the world. It is a mistake to imagine that the German brain, for instance, is constructed so differently from the American, that the German *Fuchs* can grasp niceties of the calculus which necessarily escape the American Sophomore. Nor is it logical to presume that the tasks of an engineer differ materially in the two countries. The problems to be fought out are generally speaking about the same, aside from certain minor matters which depend upon traditional systems of instruction. The battle which is being waged on German soil for the closer union and more complete understanding between mathematicians and engineers, is therefore of almost equal interest to the same two classes in America. But we must here pass over the immense amount of fruitful material, which is the product of some of the most eminent minds of Germany*—among them Felix Klein—and which throws strong light on “the necessary and sufficient amount of calculus for the engineer.”† It should be remarked, however, that the book which forms the subject of this review is produced, for use in a technical school, in the light of all this inspiring criti-

* See, e. g., the recent files of the *Jahresbericht der Deutschen Mathematiker Vereinigung*.

† Fricke, preface.

cism, and by an author whose fitness for the work possibly exceeds that of any other German mathematician.

Although the present edition is the third, the book will be reviewed as an independent work, both because no previous review has appeared in the *BULLETIN*, and because the present edition, being the first purposely prepared for use outside the Technische Hochschule in Braunschweig, is considerably revised in form and content.

The author has been considerably influenced, of course, by the special needs of his own school, since the book was originally published for use in that institution alone; and in particular every effort has been made to meet, intelligently, the exigencies of mathematical instruction in a school for engineers. We shall accept the same standpoint in criticizing the book, and shall lay stress on those parts which affect the teaching of mathematics to future engineers.

As is explicitly stated in the preface, the author seeks to maintain a reasonable standard of mathematical accuracy, but he does not desire to sacrifice to this end the simplicity of treatment and the practicability necessary to a student who intends to pursue engineering. For this reason he intentionally uses geometrical and intuitional proof when rigorous arithmetic demonstration is considered too difficult. While the reviewer believes heartily in this principle, the conjugate statement is certainly the one which needs most insistence in our American texts, for while in the past the German has been too difficult in his rigorous presentation, we in America have erred to the side of leniency. To particularize, statements should not be made which are utterly false as they stand, nor should the treatment be such as to render a further study of mathematics impossible to a student who finds himself especially drawn toward it. And so it happens that this book, designed to meet the needs of engineers, and purposely avoiding the difficulties of rigor by occasional geometrical or intuitional proof, still contains many easy methods of proof and presentation which commend themselves to our writers on the calculus, because they are generally more rigorous than those of our books, and because the results obtained are at least accurately true as stated.

The book consists roughly of four parts: (1) the differential calculus, (2) the integral calculus, (3) differential equations, (4) an appendix on functions of a complex variable; of which the

first two each consists of two sections. When we notice that all this work, though containing in its first two parts alone more than any of our elementary texts on the calculus, is compressed into 218 pages, an explanation must be made that the book is intended for use in connection with extensive lectures. The author has explained elsewhere that many examples and explanations involving the use of actual mechanical models, as well as geometrical illustrations and proofs, are given in the lectures at Braunschweig to which this book is a companion text. That the last two parts extend the field of the book far past that of our texts is apparent.

The first sixteen pages are devoted to an introduction, in which the notions of "variable," "function," "limit," "continuity," etc., are explained. The notable features of these pages is the accuracy with which these ideas are defined and explained, although the author uses geometrical ideas continually. That reasonable accuracy *can* be combined with simplicity and concreteness is best illustrated by the explanation of a continuous variable by means of the intuitional concept of motion. The word "limit" is explained in a perfectly rigorous way, which is after all simpler than the usual explanations of our texts. The fundamental character of this concept for the calculus appears fully to justify a demand for complete rigor at this point. As an application, the *existence* of the number

$$l = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n$$

is proved in a manner which can be commended even to the advanced student of mathematics as a model of simplicity and accuracy.

But the main portion of this introduction is devoted to the discussion of certain functions which are termed "elementary." These comprise : (a) the rational and irrational ("elementary algebraic") functions, which are defined as those functions which arise from x and constants by the successive application of the four elementary operations and the extraction of roots ; (b) logarithmic and exponential functions ; (c) trigonometric functions and their inverses. Practically the whole of the book is based upon these "elementary" functions, and the theorems are stated for them alone. This permits a degree of simplicity in proof and of accuracy in statement which is not

attainable when no restriction is laid upon the functions treated. Our text-books would do well to follow some such method.

The first section treats the formal rules of differential calculus. The notion of "derivative" is defined *only* for the "elementary" functions, and only at points where these functions are continuous. Perhaps the only point in the book which merits criticism is the treatment of differentials. There are two definitions given, which are unfortunately not compatible. On page 18 the differentials dx and dy are defined by the equations $dx \equiv \Delta x$, $dy = f'(x) \cdot \Delta x$, where

$$y = f(x), \quad \frac{dy}{dx} = f'(x).$$

This definition, if carefully used, is quite justifiable and logical. The other definition, on p. 17, following a usage which is unfortunately common, defines the differentials as certain "infinitesimal" quantities, which are neither zero nor finite; dx and dy being such "infinitesimal" values of Δx and Δy , respectively. As the reviewer has elsewhere remarked,* this second definition is totally illogical in a treatise which intends to use euclidean geometry, especially if intuitional ideas or ordinary theorems on continuity are to be used in the discussions. For the introduction of such "infinitesimal" quantities expressly violates the axiom of Archimedes, which is fundamental in our ordinary conceptions of continuity. The author has, however, explicitly stated on page 18 that this latter definition is not a correct one.

The derivations of the derivatives of ordinary functions are in general the same as usual. The proof of the fundamental limit,

$$\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right) = 1,$$

on page 22 is possibly worth noting, as an extremely easy, yet reasonably accurate proof.

In this first chapter the hyperbolic functions ($\sinh x$, etc.), are introduced, and their derivatives found. As this departure is made, according to the preface, at the earnest suggestion of engineers, the matter may be worth the notice of American writers for students of engineering.

* "The Doctrine of Infinity," BULLETIN, vol. 9 (2), no. 5 (Feb., 1903), p. 263.

In the next chapter a simple proof of the binomial formula for a positive integral exponent is given, based upon the relation $e^A \cdot e^B = e^{A+B}$. The proof is at least interesting in its curiousness.

The difficulties of "infinitesimal" quantities are encountered quite sharply on pages 33–35. Notice will be taken only of the non-existence (in an archimedean arithmetic) of the number ϵ of page 34, which "approaches zero continuously without becoming identically zero," and "is infinitely small." That the introduction of such quantities is in any way *necessary* to the treatment of engineering topics, or that it simplifies the study of engineering, is certainly a fallacy based solely upon familiarity and tradition.* Had Newton and Leibnitz hit upon the definition of a derivative as a *limit*, our engineers of to-day would never dream of using concepts which destroy all ordinary notions of the continuity of space and motion.

The next section considers some applications of the formal rules previously obtained. In the first chapter, on maxima and minima, it would seem that simplification might have been made; but the general method of presentation, while not differing essentially from the usual treatment, is unusually easy for a student to grasp.

The introduction of the cycloid and the extensive development of its properties in the next chapter, is noteworthy, and might well be imitated in an engineering calculus.

But the chief merit of this section consists in the third chapter, on infinite series. The treatment is reasonably (but not absolutely) rigorous, and certainly superior to the utter lack of treatment which too often obtains. The "sum" of an infinite series is defined with reasonable accuracy, the "convergence" and "absolute convergence" of series are discussed, and the condition $u_{n+1}/u_n \leq r < 1$ is established for the convergence of a series of positive terms. Even power series are discussed briefly, but without many proofs of theorems.

The average value theorem is then derived; and is used for a very simple treatment of Taylor's development, which can be recommended as simple, accurate and above all *true*. Even more remarkable, from the standpoint of comparison with the American calculus for engineers, is the investigation of the interval of convergence of the Taylor's expansions, for each of

* That this does not preclude a *proper* definition of *differentials* has already been intimated.

the "elementary" functions considered. It is clearly demonstrated that the method of such investigations, and the proof of the necessity of investigation, is within the reach of a fairly intelligent student. Last of all these comes the binomial theorem, of which very clear proof and statement are given.

The discussion of indeterminate coefficients at the end of the chapter establishes the uniqueness of McLaurin's expansion. It is remarkable to notice that the proof of the uniqueness of the expansion is just what is usually given in our texts, in place of a proof of the formula itself!

The last chapter of the section deals with some common indeterminate forms in a very acceptable manner, though the proofs are not supposed to be entirely rigorous. The use of power series as an alternate process is somewhat out of the ordinary; and the investigation of the relative intensity of the approach of x^n and $\log x$ toward infinity, is quite so, in such a text.

The second part, on integral calculus, opens with a treatment of definite and indefinite integrals which is worthy of notice and possible imitation. The accuracy of statement which characterizes the book is noticeable on page 80. The theorem that two integrals of the same integrand can differ only by an additive constant, is referred for proof to the average value theorem of differential calculus. That the author contents himself with the mere reference, and does not give the proof which he shows is necessary, is typical of the attempt at simplicity of presentation. A proof just here would probably confuse the ordinary student.

The definition of a definite integral is given as a sum (which is meant, of course, for the limit of a sum) and the connection with indefinite integrals is established by means of the average value theorem in connection with an admirable geometrical representation, which renders the reasoning concrete and therefore easy. The average value theorem for integrals is then proved and its concrete applications are not wanting in the rest of the book.

Aside from certain ordinary formulæ, the chapter further contains the interpretation of hyperbolic and trigonometric functions by means of areas, and certain formulæ for the approximate calculation of definite integrals. The first topic is of doubtful importance to engineers; but the importance of the second to them can scarcely be overestimated. Such a treat-

ment should be given, even in more detail, in all calculus texts for engineers.

In order to establish the expansion of a rational function into partial fractions, a few theorems on algebraic equations are stated in the second chapter. This enables the author to give a proof of the expansion in question which amounts to something more than a bald assertion. The Lagrange formulæ of interpolation for constructing a function of lowest possible degree taking on a given set of values for given arguments is surely of use to the engineer, and should be found, apparently, in all engineering texts on the calculus.

The remainder of the chapter establishes for the most part usual theorems. It might be remarked that the author's anxiety to introduce elliptic integrals has led him to a statement (page 111) which, without explanation, might lead a student to suppose that such an integral as

$$\int \frac{(2x-1)^2 dx}{(17-3x+6x^2-4x^3)^{\frac{1}{2}}}$$

did *not* represent an "elementary" function.

The fourth section is not especially noteworthy. The usual theorems regarding functions of several independent variables are discussed, and the treatment is not very unusual. There are, however, several minor points where the presentation is especially good, and a little more surface theory than is usual in our books is given. The proof on page 120 of the theorem

$$\frac{dy}{dx} = - \frac{\partial f / \partial x}{\partial f / \partial y},$$

where $f(x, y) = 0$ is given, is a clear instance of the necessity of considerable care in the handling of differentials, even when properly defined. Since

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

is the definition of dz (which is *otherwise* unknown) it follows that we cannot set $dz = 0$ without an assumption which practically amounts to the theorem itself! The theorem at the bottom of page 121: "Given $y = f(x_1, x_2, \dots, x_n)$, $x_i = \phi_i(x)$, then

$$\frac{dy}{dx} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dx} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dx} + \cdots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dx},$$

is really the fundamental theorem upon which all the work for functions of several variables might be based.

The third part (Section V) treats of differential equations. A single ordinary equation in one dependent variable, a set of two ordinary equations in two dependent variables, and a single partial equation in one dependent and two independent variables, are the topics treated. The selection of methods of solution is good, scarcely any well known elementary method being omitted. The best features of the section are a geometrical proof of the existence of solutions of an ordinary equation, and geometrical interpretations of the meanings of the several types of equations discussed.

The introduction of the hypergeometric series and its differential equation, at the close of the section, will sufficiently illustrate the scope of the work. Of course the reasoning is necessarily somewhat crude at points in this section, as for instance the statements that a given differential equation can always be written in normal form; but the work is generally more rigorous, and much more complete, than corresponding discussions in any elementary calculus in the English language. The 38 pages of this section compare favorably with such books as Johnson or Forsyth in giving a general oversight of the subject. In general, the short courses on differential equations given in American technical schools — when such courses are given at all — lay too much stress on ordinary equations to the total or partial exclusion of partial equations. For the problems of higher engineering very often lead to partial differential equations, and the solution of many other problems usually solved by cumbrous methods can often be effected in simple form by the use of such equations.

The short appendix on functions of a complex variable is scarcely to be mentioned for imitation in our books on calculus. Important as is the subject matter for the engineer and for the mathematical student, such work is beyond the average teacher in our technical schools, and could not be given to advantage.

For the purposes of our schools it would seem that more time should be spent on applications of the calculus to mechanics. These applications are quite as immediate as the applications to geometry, for a derivative is represented by a velocity quite as

truly and simply as by the slope of a curve; and the second derivative finds easier direct representation in acceleration than in curvature. Moreover the subject of mechanics is of more importance to the engineer than the *extended* study of curves and surfaces. Of course the geometrical interpretation of the calculus must continue to be given quite as fully, at least, as is now common; but it would certainly be useful to introduce the ideas of mechanics early in the work, and bridge the chasm between formal calculus and engineering mechanics by as many applications of the calculus to mechanics as possible.

The remarks which have been made have to do, for the most part, with the method of presentation of the calculus to future engineers in American technical schools. This is a most difficult theme, and it is scarcely hoped that the opinions which have been expressed will be generally accepted as final. Many important topics have not been discussed, since they were not suggested by the book in hand, but it is hoped that the discussions of the matters which have been touched upon will prove suggestive and possibly beneficial in the arrangement of the work in calculus in our technical institutions, and in the preparation of texts designed for such schools. The whole subject surely merits considerable careful thought and thorough discussion, particularly in view of the exceptional growth of institutions especially intended for students of engineering, and in view of present lack of any text-book on the calculus expressly designed for students of engineering which really meets their needs.

Herr Fricke's book cannot be said to be the ultimate ideal of such a text, and there are many things which would need entire revision to make the book suitable for American schools. But a great advance is marked in it over our American texts, in the combination of simplicity and concreteness with reasonable accuracy; and on this account it may well serve to suggest to the instructor or to the author of a text on the calculus, possibilities of presentation of the subject to students of engineering.

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YALE UNIVERSITY,
December, 1902.

NOTES.

THE April number (volume 4, number 2) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "The approximate determination of the form of Maclaurin's spheroid," by G. H. DARWIN; "On twisted cubic curves that have a directrix," by H. S. WHITE; "Ueber Curvenintegrale im m -dimensionalen Raum," by L. HEFFTER; "The generalized Beltrami problem concerning geodesic representation," by E. KASNER; "On the holomorph of a cyclic group," by G. A. MILLER; "Quadric surfaces in hyperbolic space," by J. L. COOLIDGE; "Ueber die Reducibilität der reellen Gruppen linearer homogener Substitutionen," by A. LOEWY; "On the possibility of differentiating term by term the developments for an arbitrary function of one real variable in terms of Bessel functions," by W. B. FORD; "On a certain congruence associated with a given ruled surface," by E. J. WILCZYNSKI; "On the class number of the cyclotomic number field $k(e^{2\pi i/p^n})$," by J. WESTLUND.

THE San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY will hold its third regular meeting at Stanford University, on Saturday, April 25, 1903.

THE twenty-fifth anniversary of the founding of the Yale Mathematical Club was celebrated on January 20th by a dinner given by Mr. George E. Dimock, at which the members of the Yale mathematical faculty, graduate students and several invited guests were present. Professor Phillips acted as toastmaster and responses were made by ex-President Dwight, President Hadley, Professors Gibbs, Pierpont, Smith and others. The club was founded by Professor Gibbs, and among the more important papers presented at its meetings have been Professor H. A. Newton's later researches on meteors, and Professor Gibbs's development of his vector analysis. Several facts of historical interest to the department were recalled, among which the following may be of general interest. In 1714 the first (recorded) mathematical books were placed in the Yale College library, among which were copies of Newton's *Optics*, and the second edition of his *Principia* which were presented to the college

by Sir Isaac Newton himself. The acquisition of these and other books evidently stimulated the study of mathematics, and previous to 1866 courses in conic sections and fluxions were given at Yale, which was the first college in America to include higher mathematics in its curriculum. It was also recalled that before 1860 no less than three Yale instructors, Professors Stanley, Loomis, and Newton availed themselves of extended periods of study in France and England, being among the pioneer American mathematicians in foreign study.

THE next meeting of the Deutsche Mathematiker-Vereinigung will be held, in connection with the seventy-fifth meeting of the association of German naturalists and physicians, at Cassel, September 20-26, 1903. Abelian functions and theoretical mechanics are especially proposed as centers of discussion at this meeting.

THE several German universities below offer during the summer semester of the current academic year courses in mathematics as follows :

UNIVERSITY OF BONN.—By Professor H. KORTUM : Exercises in the mathematical seminar, two hours ; Infinite series, two hours ; Elements of differential and integral calculus, four hours.—By Professor R. LIPSCHITZ : Exercises in the mathematical seminar, two hours ; Application of the infinitesimal calculus to the theory of space, four hours.

UNIVERSITY OF BRESLAU.—By Professor J. ROSANES : Exercises in the mathematical seminar, one hour ; Analytic geometry of the plane, four hours ; Elements of the theory of determinants, two hours.—By Professor R. STURM : Exercises in the mathematical seminar, two hours ; Theory of geometric transformations, part I, four hours ; Descriptive geometry and graphic statics, three hours. By Professor F. LONDON : Theory of definite integrals and of Fourier's series, three hours.

UNIVERSITY OF ERLANGEN.—By Professor P. GORDAN : Geometry of space, four hours ; Theory of numbers, four hours ; Exercises in the seminar, three hours.—By Professor M. NOETHER : Differential and integral calculus, II, four hours ; Geometric and analytic exercises.

UNIVERSITY OF FREIBURG.—By Professor J. LÜROTH : Higher analytic geometry of curves and surfaces, four hours ; Theoretic astronomy, three hours.—By Professor L. STICKEL-

BERGER : Integral calculus, five hours ; Elliptic functions, three hours ; Mathematical seminar. By Professor A. LOEWY : Theory of algebraic equations, four hours ; On the foundations of geometry, two hours.

UNIVERSITY OF GIESSEN.—By Professor M. PASCH : Analytic geometry of the plane, four hours ; General expedients in the theory of functions, two hours ; Exercises in the mathematical seminar, one hour.—By Professor E. NETTO : Introduction to algebra, four hours ; Differential equations, two hours ; Exercises in the mathematical seminar, one hour.—By Professor J. WELLSTEIN : Descriptive geometry, second part, with exercises, six hours ; Introduction to the geometry of position, two hours ; Arithmetic theory of forms, two hours.

UNIVERSITY OF GÖTTINGEN.—By Professor FELIX KLEIN : Encyclopædia of geometry, four hours ; Mathematical seminar, two hours.—By Professor D. HILBERT : Differential equations, four hours ; Mechanics of continua, two hours ; Exercises in the mathematico-physical seminar, two hours.—By Professor M. BRENDL : Insurance, two hours ; Exercises in the integration of differential equations, two hours ; Special perturbations, two hours ; Mathematical exercises in the insurance seminar, two hours.—By Professor F. SCHILLING : Differential and integral calculus, I, four hours ; Graphic statics, one hour ; Exercises in graphic statics, two hours.—By Dr. E. ZERMELO : Analytic geometry, four hours ; Exercises in the integration of differential equations, two hours.—By Dr. O. BLUMENTHAL : Elliptic and modular functions, three hours ; Exercises in the integration of differential equations, two hours.

UNIVERSITY OF HALLE-WITTENBERG.—By Professor G. CANTOR : Differential and integral calculus, five hours ; Exercises in the mathematical seminar, one hour.—By Professor A. WANGERIN : Differential equations, four hours ; Analytic geometry of the plane, three hours ; Spherical trigonometry and mathematical geography, three hours ; Exercises in the mathematical seminar, one hour. By Professor H. GRASSMANN : Application of descriptive geometry to surfaces of the second degree, with exercises, two hours.—By Professor H. BUCHHOLZ : Probabilities and the method of least squares, two hours ; Practical exercises in determination of geographic position, for mathematicians and geographers, two hours.—By Dr. F. BERNSTEIN :

Foundations of geometry, two hours ; Exercises in the foundations of geometry, one hour.

UNIVERSITY OF HEIDELBERG.—By Professor L. KÖNIGSBERGER : Differential and integral calculus, four hours ; Theory of functions, four hours ; Seminar, two hours.—By Professor M. CANTOR : Application of analysis to higher algebraic plane curves, four hours ; Arithmetic and algebra, three hours.—By Professor F. POCKELS : Partial differential equations of physics, two hours.—By Professor F. EISENLOHR : Probabilities, three hours.—By Professor K. KOEHLER : Plane analytic geometry, three hours.—By Professor G. LANDSBERG : Theory of determinants and invariants, four hours ; Selected chapters in the theory of algebraic functions, two hours.—By Dr. K. BOEHM : Theory of elliptic functions.

UNIVERSITY OF JENA.—By Professor J. THOMAE : Plane analytic geometry, four hours ; Mathematical geography, four hours.—By Professor A. GUTZMER : Differential calculus, with exercises, five hours ; Introduction to the theory of differential equations, five hours.—By Professor G. FREGE : Riemann's theory of functions, four hours ; Mathematical exercises, two hours.

UNIVERSITY OF KIEL.—By Professor L. POCHHAMMER : Plane analytic geometry, four hours ; Theory of definite integrals, four hours ; Introduction to probabilities, one hour ; Exercises in the mathematical seminar, one hour.—By Professor P. STÄCKEL : Differential calculus and introduction to analysis, four hours ; Curvature of surfaces, one hour ; Analytic mechanics, four hours ; Exercises in the mathematical seminar, on mechanics, one hour.—By Professor E. KOBOLD : Elementary geodesy, two hours ; Exercises in geodesy, one hour.

UNIVERSITY OF LEIPSIK.—By Professor K. NEUMANN : Differential and integral calculus, three hours ; Mathematical seminar, two hours.—By Professor A. MAYER : Ordinary differential equations, four hours, with exercises.—By Professor O. HÖLDER : Applications of elliptic functions, three hours ; Projective geometry treated synthetically, three hours ; Mathematical seminar, one hour.—By Professor F. ENGEL : Analytic geometry, four hours ; Theory of transformation groups, two hours ; Algebraic equations, two hours ; Mathematical seminar, one hour.—By Professor F. HAUSDORFF : Differential geometry, four hours ; Exercises, one hour.

UNIVERSITY OF MARBURG.—By Professor E. HESS: Plane geometry treated analytically and synthetically, four hours; Selected chapters from higher analysis, four hours; Exercises in the mathematical seminar, three hours.—By Professor K. HENSEL: Differential calculus, five hours; Theory of algebraic functions of a single variable and their application to the theory of algebraic curves and of the abelian integrals, four hours; Seminar, two hours.—By Dr. F. VON DALWIGK: Theory of functions, five hours; Geodesy, two hours.—By Dr. H. JUNG: Theory of numbers, four hours; Algebra, II, two hours.

UNIVERSITY OF ROSTOCK.—By Professor O. STAUDE: Differential and integral calculus, four hours; Seminar, two hours.

UNIVERSITY OF STRASSBURG.—By Professor T. REYE: Selected chapters from the higher synthetic geometry, three hours; Theory of potential, three hours; Exercises in the seminar, two hours.—By Professor H. WEBER: Definite integrals and the introduction to the theory of functions, four hours; Higher algebra, four hours; Exercises in the seminar, two hours. By Professor G. ROTH: Differential and integral calculus, three hours; Exercises in differential and integral calculus, two hours; Plane analytic geometry, three hours.—By Professor M. DISTELI: Analytic geometry of space, three hours; Descriptive geometry, II, two hours; Exercises in descriptive geometry, four hours; Exercises in the seminar, two hours.

UNIVERSITY OF TÜBINGEN.—By Professor A. VON BRILL: Analytic geometry of space, three hours; Theory of curvature of surfaces, four hours; Exercises in the seminar, two hours.—By Professor H. STAHL: Elementary analysis, three hours; Higher analysis, three hours; Exercises in each, one hour.—By Professor L. MAURER: Descriptive geometry, two hours; One-valued functions of a complex variable, two hours; Exercises in descriptive geometry, two hours; Exercises in the theory of functions, one hour.

THE February number of the *Popular Science Monthly* contains a statistical study by Professor J. MCK. CATTELL of eminent men of the past. Professor Cattell's list of a thousand names, selected according to the space devoted to them in six representative encyclopædias and biographical dictionaries, includes the following mathematicians: Newton (14th), Descartes

(23d), Leibnitz (34th), Pascal, Pythagoras, Napier, Laplace, Lagrange, Huygens, Archimedes, Cardan, Dupin, Euclid, Euler, Monge, Gauss, Fermat. The order of the names is based of course on general, not mathematical, reputation.

THE Mathematical society of France has elected as its president for 1903, Professor P. PAINLEVÉ. The four new members of the council are MM. KOENIGS, LECORNU, R. PERRIN, and TOUCHE. Professor G. MITTAG-LEFFLER has been made an honorary member, the second foreign member of the Society, the other being Professor L. CREMONA.

PROFESSOR H. POINCARÉ has been elected a corresponding member of the Brussels academy.

PROFESSOR P. APPELL has been elected dean of the Paris faculty of sciences, succeeding Professor G. DARBOUX, who has been made honorary dean.

PROFESSOR JOSEPH LARMOR, fellow of St. John's College, Cambridge, and secretary of the Royal society, has been elected Lucasian professor of mathematics, succeeding the late Sir George Gabriel Stokes.

IT is proposed by Cambridge University to secure an appropriate memorial of the late Sir G. G. STOKES. For this purpose a committee has been appointed consisting of the chancellor, vice-chancellor, Professors Jebb, Forsyth, Darwin, Ball, Thomson, and W. Burnside.

PROFESSOR J. J. THOMSON, of Cambridge, England, will deliver at Yale University, beginning May 14, a course of eight lectures on "Recent developments in our ideas of electricity." These lectures are the first to be given on the Silliman foundation at Yale.

IN the investigation undertaken by the aid of the Carnegie Institution into certain phases of geophysical problems common to mathematics, astronomy, physics, chemistry and geology, mathematics is to be represented by Professors C. S. SLICHTER, of the University of Wisconsin, and L. M. HOSKINS, of Leland Stanford University.

DR. M. B. PORTER has resigned his position as assistant professor of mathematics in the academic department of Yale

University, to accept the professorship of mathematics recently offered him at the University of Texas, where he will have full charge of the department of mathematics.

DR. G. A. MILLER has been promoted from an assistant to an associate professorship of mathematics at Stanford University.

DR. F. R. MOULTON has been promoted to an assistant professorship of mathematics at the University of Chicago.

THE deaths are announced of Professor F. J. STUDNICKA, of the Bohemian University of Prague, and of Professor ANTON PUCHTA, of the University of Czernowitz.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AKULOV (L. A.). See PERRY (J.).

BALL (W. W. R.). Breve compendio di storia delle matematiche. Versione dall'inglese, con note, aggiunte e modificazioni di D. Gambioli e G. Puliti, riveduta e corretta da G. Loria. Vol. I: Le matematiche dall'antichità al rinascimento. Versione di G. Puliti, con una nota originale su la scuola pitagorica. Bologna, Zanichelli, 1903. 8vo. 10 + 284 pp. Fr. 8.00

BAZELINSKY (V. V.). See PERRY (J.).

BÖCHER (M.). See BRIGGS (G. R.).

BOREL (E.). Leçons sur les fonctions méromorphes, professées au Collège de France, recueillies et rédigées par L. Zoretti. Paris, Gauthier-Villars, 1903. 8vo. 6 + 124 pp. (Nouvelles leçons sur la théorie des fonctions, 4e partie.) Fr. 3.50

BRIGGS (G. R.). The elements of plane analytic geometry: a text-book, including numerous examples and applications, and especially designed for beginners. 7th edition, revised and enlarged by M. Böcher. New York, Wiley, 1903. 12mo. 5 + 191 pp. Cloth. \$1.00

CATALOG mathematischer Modelle für den höheren mathematischen Unterricht, veröffentlicht durch die Verlagshandlung von Martin Schilling in Halle a. S. 6te Auflage. Halle, 1903. 8vo. 130 pp. Cloth. M. 1.00

CHARASOFF (G.). Arithmetische Untersuchungen über Irreduktibilität. (Diss.) Heidelberg, 1902. 8vo. 67 pp.

DU BOBERIL (R.). Pascal et Riemann. Paris, Dubois, 1902. 18mo. 22 pp.

FORMULARIO di geometria analitica. Torino, Bertero, 1902. 8vo. 14 pp.

GAMBIOLI (D.). See BALL (W. W. R.).

- GERLACH (A.). Ueber die Anwendbarkeit der Methode des arithmetischen Mittels auf eine von zwei konfokalen Ellipsen begrenzte Ringfläche. (Diss., Leipzig.) Frankfurt a. M., Knauer, 1902. 8vo. 32 pp., 1 plate. M. 0.80
- HADAMARD (J.). Note sur l'induction et la généralisation en mathématiques. Paris, Colin, 1903. 8vo. pp. 441-444. (Bibliothèque du Congrès international de philosophie, Vol. III.)
- . Sur certaines surfaces minima. 8vo. 3 pp. (*Bulletin des Sciences Mathématiques*, (2) Vol. 26, Dec. 1902.)
- . Sur les réseaux de coniques. 8vo. 4 pp. (*Bulletin des Sciences Mathématiques*, (2) Vol. 25, Jan. 1901.)
- HEIBERG (J. L.). See PTOLEMAEUS (C.).
- HERO ALEXANDRINUS. Opera quae supersunt omnia. Volumen III: Vermessungslehre und Dioptra. Griechisch und deutsch von H. Schöne. Leipzig, 1903. 8vo. 21 + 366 pp. M. 8.00
- LORIA (G.). See BALL (W. W. R.).
- LÜTKEMEYER (G.). Ueber den analytischen Charakter der Integrale von partiellen Differentialgleichungen. (Diss.) Göttingen, 1902. 8vo. 48 pp.
- MARTIN (E. N.). On the imprimitive substitution groups of degree fifteen and the primitive substitution groups of degree eighteen. Dissertation presented to the faculty of Bryn Mawr College for the degree of Doctor of Philosophy. Baltimore, Friedenwald, 1903.
- PEANO (G.). Aritmetica generale e algebra elementare. Turin, Paravia, 1902. 8vo. 144 pp. Fr. 2.40
- PERRY (J.). The calculus for engineers; translated by L. A. Akulov and V. V. Bazhinsky. (In 2 volumes.) Part I. St. Petersburg, 1902. 8vo. 330 pp. (Russian.) M. 9.00
- PTOLEMAEUS (C.). Opera quae exstant omnia. Edidit J. L. Heiberg. Volumen I: Syntaxis mathematica. Pars II: libri 7-13. Leipzig, 1903. 8vo. 4 + 608 pp. M. 12.00
- PULITI (G.). See BALL (W. W. R.).
- SCHÖNE (H.). See HERO ALEXANDRINUS.
- ZORETTI (L.). See BOREL (E.).
- VIVANTI (G.). Complementi di matematica ad uso dei chimici e dei naturalisti. Milano, Hoepli, 1903. 16mo. 10 + 381 pp. (Manuali Hoepli.)

II. ELEMENTARY MATHEMATICS.

- ALLCOCK (C. H.). Theoretical geometry for beginners. London, Macmillan, 1903. 12mo. 146 pp. Cloth. \$0.40
- ANDREW (S. O.). Geometry: elementary treatise on theory and practice of Euclid, having in view new regulations of Oxford and Cambridge local, London matriculation, Board of Education, and other examinations. London, Murray, 1903. 12mo. 194 pp. Cloth. 2s.
- BAKER (W. M.) and BOURNE (A. A.). Elementary geometry, books 1-4. London, Bell, 1903. 8vo. 304 pp. Cloth. 3s.

—. Elementary geometry, books 1 and 2. 3d edition, revised. London, Bell, 1903. 8vo. 158 pp. Cloth. 1s. 6d.

BOURNE (A. A.). See BAKER (W. M.).

BRITISH ASSOCIATION Meeting in Glasgow, 1901. Discussion on the teaching of mathematics which took place on September 14th, at a joint meeting of two sections: Section A, Mathematics and Physics; Section L, Education. Chairman of the joint meeting: the right hon. Sir John E. Gorst, K.C., M.P., President of Section L. Edited by Professor Perry. To which is now added the Report of the British Association Committee drawn up by the Chairman, Professor Forsyth. London, Macmillan, 1902. 12mo. 10 + 123 pp. Cloth.

DICKNETHER (F.). Lehrbuch der Arithmetik nebst Uebungsaufgaben für Mittelschulen. Teil 2. München, Lindauer, 1903. 8vo. 4 + 111 pp. M. 1.40

DIMBARRE (H.). Cours de trigonométrie rectiligne. Marseille, Laffitte, 1903. 4to. 262 pp.

EDWARDS (R. W. K.). Elementary plane and solid mensuration. London, 1902. 8vo. 334 pp. Cloth. 3s. 6d.

EGGAR (W. D.). Practical exercises in geometry. London, Macmillan, 1903. 12mo. 300 pp. Cloth. \$0.60

ELEMENTOS de aritmética, con algunas nociones de algebra; por los Hermanos de las escuelas cristianas. 6a edicion, correspondiente a los cursos medio y superior. Paris, Mame, 1903. 16mo. 392 pp.

EYSSÉRIC et PASCAL. Eléments de trigonométrie rectiligne, à l'usage des élèves de l'enseignement secondaire et des candidats au baccalauréats. 21e édition. Paris, Delagrave, 1902. 12mo. 190 pp.

FINKEL (B. F.). Mathematical solution book. 4th edition, revised and enlarged. Springfield, Mo., Kibler, 1903. 8vo. 549 pp., 14 portraits. Cloth. \$2.00

FORSYTH (A. R.). See BRITISH ASSOCIATION.

FUCINI (C.). Geometria plana e nozioni di geometria solida per le scuole secondarie inferiori. 6a edizione. Genova, Tipografia della Gioventù, 1903. 8vo. 126 pp. Fr. 1.50

—. Geometria plana per gl'istituti nautici. 6a edizione. Genova, Tipografia della Gioventù, 1903. 8vo. 111 pp. Fr. 1.50

GAJDECZKA (J.). Maturitäts-Prüfungsfragen aus der Mathematik, zusammengestellt und mit Auflösungen versehen. Wien, Deuticke, 1903. 8vo. 4 + 74 pp. M. 1.20

GRÉVY (A.). Eléments d'algèbre, à l'usage des élèves des classes de troisième à première A et B (programmes du 31 mai 1902). Paris, Nony, 1903. 18mo. 196 pp. Fr. 1.75

—. Géométrie (compléments), à l'usage des élèves des classes de seconde et première C et D (programmes du 31 mai 1902). Paris, Nony, 1903. 18mo. 120 pp. Fr. 1.25

GUGLIELMI (A.). Nozioni di algebra per le scuole tecniche e normali, con molti esempi ed esercizi e due note. Napoli, Romano, 1902. 16mo. 76 pp. Fr. 0.80

- LALANDE (J. DE). Tables de logarithmes à cinq décimales pour les nombres et pour les sinus; revues par Reynaud. Edition stéréotype, augmentée de formules pour la résolution des triangles. Paris, Gauthier-Villars, 1903. 18mo. 42 + 236 pp. Fr. 2.00
- Tables de logarithmes, étendues à sept décimales par F. C. M. Marie. Précédées d'une instruction dans laquelle on fait connaître les limites des erreurs qui peuvent résulter de l'emploi des logarithmes des nombres et des lignes trigonométriques, par Reynaud. Nouvelle édition, augmentée de formules pour la résolution des triangles. Paris, Gauthier-Villars, 1903. 16mo. 42 + 238 pp. Fr. 3.50
- LEBON (E.). Géométrie appliquée (rédigée conformément au programme des écoles normales primaires et du brevet supérieur), comprenant des notions de trigonométrie rectiligne, le levé des plans, etc. 3e édition, revue et augmentée. Paris, Delalain, 1903. 16mo. 8 + 41 + 248 pp. Fr. 3.50
- LIEBER (H.) und LÜHMANN (F. VON). Leitfaden der Elementarmathematik. Nach den Bestimmungen der preussischen Lehrpläne vom Jahre 1901 neu bearbeitet von C. Müsebeck. Teil 2: Arithmetik. Ausgabe B für Realschulen, Progymnasien und Realprogymnasien. Berlin, Simion, 1902. 8vo. 4 + 92 pp. Boards. M. 1.00
- LONDON matriculation mathematics. Guide to the mathematics of the matriculation examination of London University. London, Clive, 1903. 8vo. 152 pp. Half leather. (University tutorial series.) 1s. 6d.
- LÜHMANN (F. VON). See LIEBER (H.).
- MACÉ DE LÉPINAY (A.). See VACQUANT (C.).
- MARIE (F. C. M.). See LALANDE (J.).
- MARTINAUD (M. A.). Eléments d'arithmétique et de géométrie, à l'usage de l'enseignement secondaire, rédigés conformément aux programmes officiels de 1902. Premier cycle. Paris, Belin, 1902. 12mo. 207 pp.
- MARTUS (H. D. E.). Maxima und Minima; ein geometrisches und algebraisches Übungsbuch. 2ter Abdruck. Hamburg, 1903. 8vo. M. 1.80
- Mathematische Aufgaben zum Gebrauche in den obersten Klassen höherer Lehranstalten. (In 4 Teilen.) Teil I. 11te Auflage. Dresden, 1903. 8vo. Cloth. M. 4.00
- MAYER (J. E.). Das mathematische Pensum des Primaners; ein Hilfsbuch für den Primaner humanistischer und realistischer Gymnasien. Heft 2: Kettenbrüche, Teilbruchreihen, diophantische Gleichungen, Stereometrie. Freiburg i. B., Lorenz, 1902. 8vo. 53 pp. M. 1.00
- MIGLIACCI (R.). Una nuova dimostrazione al teorema di Pitagora. Livorno, Giusti, 1902. 8vo. 6 pp.
- MÜSEBECK (C.). See LIEBER (H.).
- PANIZZA (F.). Aritmetica razionale. 4ta edizione riveduta. Milano, Hoepli, 1903. 16mo. 10 + 210 pp. (Manuali Hoepli.)
- PASCAL. See EYSSÉRIC.
- PERRY (J.). See BRITISH ASSOCIATION.

- PINCHERLE (S.). *Geometria metrica e trigonometria*. 6a edizione. Milano, Hoepli, 1903. 16mo. 4 + 160 pp. (Manuali Hoepli.)
- REYNAUD. See LALANDE (J. DE).
- SCHURIG (R.). *Katechismus der Algebra*. 5te Auflage. Leipzig, Weber, 1903. 12mo. 7 + 236 pp. Cloth. (Weber's illustrierte Katechismen, No. 71.) M. 3.00
- SCOTTI (G.). *Elementi di geometria intuitiva ad uso del ginnasio inferiore e dei corsi complementari secondo gli ultimi programmi governativi*. 3a edizione. Torino, Tipografia Salesiana, 1903. 8vo. 139 pp. Fr. 1.00
- . *Elementi di geometria ad uso del ginnasio superiore secondo gli ultimi programmi governativi*. 3a edizione. Torino, Tipografia Salesiana, 1903. 8vo. 128 pp. Fr. 1.50
- SKOBCZYK (F.). *Leitfaden der Geometrie für Präparandenanstalten und Seminare*. Teil I: Planimetrie. Halle, Schroedel, 1903. 8vo. 8 + 144 pp. M. 1.75
- SOCCHI (A.) e TOLOMEI (G.). *Aritmetica generale e algebra; libro di testo per la terza classe del liceo, conforme ai vigenti programmi*. Firenze, Le Monnier, 1903. 16mo. 128 pp. Fr. 1.50
- TOLOMEI (G.). See SOCCI (A.).
- VACQUANT (C.) et MACÉ DE LÉPINAY (A.). *Principes d'algèbre à l'usage des élèves de l'enseignement scientifique (premier cycle, classes de sixième, cinquième, quatrième et troisième; deuxième cycle, classes de seconde et de première)*. 15e édition conforme aux programmes du 31 mai 1902. Paris, Delagrave, [1903]. 18mo. 524 pp.
- WENTWORTH (G. A.). *Plane trigonometry*. 2d revised edition. Boston, Ginn, 1903. 12mo. 6 + 141 + 21 pp. Cloth. \$0.75

III. APPLIED MATHEMATICS.

- BOBYLEV (D.). *Supplement to the course in analytical mechanics*. St. Petersburg, 1903. 4to. 106 pp. (Russian.)
- BUCHETTI (J.). *De la mesure du travail des machines-outils, etc.; Dynamos et dynamomètres*. Paris, Béranger, 1903. 8vo. 79 pp., 17 plates.
- BUNGERS (E.). *Ueber die Bewegung eines schweren Punktes auf einem Kegelschnitt, der mit konstanter Geschwindigkeit um seine vertikale Hauptachse rotiert*. (Diss., Halle.) Halle a. S., Kaemmerer, 1902. 8vo. 85 pp., 2 plates. M. 1.00
- CLERKE (A. M.). *Problems in astrophysics*. London and New York, Macmillan, 1903. 8vo. 16 + 567 pp. Cloth. \$6.00
- DURLEY (R. J.). *Kinematics of machines: an elementary text-book*. New York, Wiley, 1903. 8vo. 8 + 379 pp. Cloth. \$4.00
- EDSER (E.). *Light for students*. London, Macmillan, 1903. 12mo. 8 + 579 pp. Cloth. \$1.50
- FELDBLUM (M.). *Geometria wykreslna*. Warsaw, 1902. 8vo. 16 + 325 pp. M. 6.00
- FELDMANN (C.). *Asynchrone Generatoren für ein- und mehrfache Wechselströme; ihre Theorie und Wirkungsweise*. Berlin, Springer, 1903. 8vo. 4 + 134 pp. M. 3.00

- FLEMING (J. A.). Waves and ripples in water, air, and aether. London, 1902. 8vo. 312 pp. Cloth. 5s.
- KESSLER (J.). Die Dampfmaschinen. Abteilung 2: Berechnung der Dampfmaschinen; kurzgefasste Theorie der Wärme, der Gase und des Wasserdampfes; Theorie der Dampfmaschinen und Anleitung zur Berechnung derselben. 2te, vermehrte und verbesserte Auflage. Mit zahlreichen Rechnungsbeispielen. Hildburghausen, Pezoldt, 1903. 8vo. 59 pp. (Technische Lehrhefte, Abteilung B: Maschinenbau, Heft 6a.) M. 1.80
- KRAETZER (A.). See ZIPP (H.).
- KÜBLER (J.). Die Berechnung der Kessel- und Gefäßwandungen. Teil I: Aufstellung der allgemeinen Gleichungen. Mit einem Anhang: Welches Hindernis versperrt in der Knicktheorie den Weg zur richtigen Erkenntnis? Leipzig, Teubner, 1902. 8vo. 52. M. 1.60
- LAISANT (C. A.). Analogies entre les courbes funiculaires et les trajectoires d'un point mobile. 8vo. 6 pp. (*Nouvelles Annales de Mathématiques.*)
- LONDON matriculation model answers: mechanics. Papers from June 1890 to June 1898, and for September, 1902. (University Tutorial Series.) London, Clive, 1903. 8vo. 128 pp. 2s.
- MIE (G.). Die neueren Forschungen über Ionen und Elektronen. Stuttgart, Enke, 1903. 8vo. 40 pp. (Sammlung elektrotechnischer Vorträge.) M. 1.20
- NEUMANN (J.). Die Versicherung mit Gewinn-Antheil bei den Lebensversicherungs-Gesellschaften des Deutschen Reiches, nebst tabellarischen Uebersichten zur Vergleichung des Vermögens- und Geschäftsstandes Ende 1901, sowie der Prämien für die wichtigsten Versicherungsformen. Berlin, 1903. 12mo. 5 + 231 pp., 2 plates. Cloth. M. 2.00
- RAKHMILOVICH (E. G.). Brief treatise of statistics. 3d edition. Odessa, 1902. 8vo. 252 pp. (Russian.) M. 5.00
- ROTH (P.). Die Festigkeitstheorien und die von ihnen abhängigen Formeln des Maschinenbaues. (Diss.) Berlin, 1902. 8vo. 45 pp.
- SCHOUTE (J. C.). Die Stelär-Theorie. Jena, Fischer, 1903. 8vo. 4 + 175 pp. M. 3.00
- SCHULTZ (E.). 333 Aufgaben allgemeinen und praktischen Inhaltes aus dem Gebiete der Körperberechnung; eine Ergänzung zu jedem Lehrbuch der Stereometrie. Essen, Baedeker, 1903. 8vo. 4 + 48 pp. M. 0.80
- VOSS (R. von). Grundzüge der Gleichstromtechnik. Als Lehrbuch beim Unterricht an technischen Fachschulen sowie als Hilfsbuch für Studierende höherer technischer Lehranstalten bearbeitet. Teil 1. Hildburghausen, Pezoldt, 1903. 8vo. 7 + 96 pp., 2 plates. (Technische Lehrhefte, Abteilung B: Maschinenbau, Heft 13.) M. 3.00
- ZIPP (H.). Berechnung elektrischer Leitungsnetze, mit specieller Berücksichtigung der Hochspannung. Herausgegeben von A. Kraetzer. In 2 Teilen. Berlin, 1902. 8vo. 84 + 96 pp. Cloth. M. 3.60

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Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

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Articles for insertion in the BULLETIN should be addressed to the
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501 West 116th Street, New York City.

The following dates have been fixed for the meetings of the Society:

A. M.	A. M.
Sat., Feb. 28, 1903, 11:00	Sat., Oct. 31, 1903, 11:00
" April 25, " "	M.-Tu., Dec 28-29, " "

The Chicago Section of the Society will meet at Northwestern University on Saturday, April 11, 1903.

The San Francisco Section will meet at Stanford University on Saturday, April 25, 1903.

The Tenth Summer Meeting and Fourth Colloquium of the Society will be held at the Massachusetts Institute of Technology, Boston, Mass., during the week beginning August 31, 1903.

BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

CONTINUATION OF THE BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

F. N. COLE A. ZIWET D. E. SMITH F. MORLEY

2ND SERIES, VOL. IX., No. 9

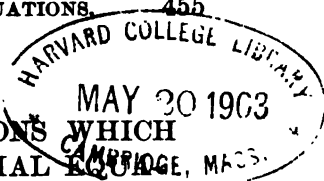
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SINGULAR POINTS OF FUNCTIONS WHICH SATISFY PARTIAL DIFFERENTIAL EQUATIONS OF THE ELLIPTIC TYPE.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society, December 30, 1902.)

IN the study of the nature of isolated singular points of harmonic functions of two variables * the following theorem may well be given a fundamental place :

I. *If the harmonic function u becomes infinite for every method of approaching the isolated singular point (x_0, y_0) , then u has the form*

$$(1) \quad C \log \sqrt{(x-x_0)^2 + (y-y_0)^2} + v(x, y),$$

where C is a constant and v is harmonic at (x_0, y_0) .

This theorem follows at once from well known facts concerning functions of a complex variable.† It is, however, highly desirable to obtain some other proof for it in order to be able to follow out consistently the method introduced by Riemann of deducing the theory of functions of a complex variable from the theory of harmonic functions of two real variables. Such a proof I have recently found, and it turns out that it can be at once applied to large classes of partial differential equations which include Laplace's equation in two dimensions as a very special case.

The theorem thus generalized, together with some applications, forms the subject of the present paper.

* I speak of a function of the n variables $x_1, \dots, x_n$ as harmonic at the point $(a_1, \dots, a_n)$ if throughout the neighborhood of this point it has continuous partial derivatives of the first two orders and satisfies Laplace's equation $\sum \partial^2 u / \partial x_i^2 = 0$. I speak of it as harmonic throughout a region if it is harmonic at every point of the region. By an isolated singular point of a harmonic function I understand a point at which it fails to be harmonic, although it is harmonic at every other point in the neighborhood of this point.

† Cf. *Annals of Mathematics*, Second Series, Vol. I (1899), p. 38. The proof can be given most readily by noticing that the derivative of the function of the complex variable $x + yi$ of which u is the real part is single valued in the neighborhood of the point $x_0 + y_0 i$ and can therefore be developed about this point by Laurent's theorem. Integrating this series we have a development for the function of which u is the real part from which the theorem follows without difficulty.

1. *Laplace's Equation in n Dimensions.*

We shall consider in this section a harmonic function $u(x_1, \dots, x_n)$ of n independent variables. If $n > 2$, theorem I takes the form

II. *If the harmonic function u becomes infinite for every method of approaching the isolated singular point $(\bar{x}_1, \dots, \bar{x}_n)$, then u has the form*

$$(2) \quad C \left[\sum_{k=1}^{k=n} (x_k - \bar{x}_k)^2 \right]^{\frac{2-n}{2}} + v(x_1, \dots, x_n),$$

where C is a constant and v is harmonic at $(\bar{x}_1, \dots, \bar{x}_n)$.

For the sake of simplicity of exposition we will prove this theorem for the case $n = 3$, but the proof for any larger value of n ,* and also for the case $n = 2$ (Theorem I) is practically identical with that here given.

We suppose, then, that $u(x, y, z)$ is harmonic throughout the neighborhood of $P_0 = (x_0, y_0, z_0)$ and becomes infinite for every method of approaching P_0 . Describe a sphere S about P_0 as centre small enough so that u is harmonic everywhere within and on the surface of S except at P_0 , and let $\bar{u}$ be the function harmonic at every point within and on S and having on S the same values as u . The function

$$(3) \quad F(x, y, z) = u - \bar{u}$$

is then harmonic at every point within and on S except at P_0 , vanishes at every point on S , and becomes infinite for every method of approaching P_0 .

Let $P_1 = (x_1, y_1, z_1)$ be any point within S other than P_0 , and let $G(x, y, z)$, or more explicitly $G(x, y, z; x_1, y_1, z_1)$, be the Green's function which becomes infinite at this point like $1/r$ but is otherwise harmonic within and on S and vanishes on S .

Let us now consider the surface $F(x, y, z) = c_0$ where c_0 is a numerically large constant which we take to be positive or negative according as F becomes positively or negatively infinite at P_0 . This surface, which we will denote by S_0 , is a closed analytic surface surrounding P_0 and lying wholly within

* That all the facts concerning Laplace's equation of which we are about to make use admit of immediate generalization to space of n dimensions is well known. Cf. Kronecker: Vorlesungen über die Theorie der einfachen und der vielfachen Integrale, Leipzig, 1894. In Vorlesungen 16 and 17, proofs of many of the theorems are given.

S , and within S_0 the function F is numerically greater than c_0 . Accordingly if we denote by n the interior normal to this surface, it is clear that $\partial F/\partial n$ does not change sign on the surface, being nowhere negative if F becomes positively infinite at P_0 , nowhere positive if F becomes negatively infinite.

Similarly let c_1 be a large positive constant and consider the surface (which as it happens will in this case be a sphere) $G(x, y, z) = c_1$. This surface, which we will call S_1 , lies within S , surrounds P_1 , and on it $\partial G/\partial n$ does not change sign. We suppose the absolute values of c_0 and c_1 to be so great that S_0 and S_1 lie wholly outside of one another.

Now apply Green's theorem to the region bounded by S , S_0 , and S_1 . Since F and G both vanish on S this gives

$$(4) \quad \int_{S_1} \left(F \frac{\partial G}{\partial n} - G \frac{\partial F}{\partial n} \right) dS = \int_{S_0} \left(G \frac{\partial F}{\partial n} - F \frac{\partial G}{\partial n} \right) dS.$$

Accordingly, by applying the law of the mean for integrals, we have

$$(5) \quad F_{S_1} \int_{S_1} \frac{\partial G}{\partial n} dS - c_1 \int_{S_1} \frac{\partial F}{\partial n} dS = G_{S_0} \int_{S_0} \frac{\partial F}{\partial n} dS - c_0 \int_{S_0} \frac{\partial G}{\partial n} dS$$

where G_{S_0} and F_{S_1} denote the values which the functions G and F take on at certain points of the surfaces S_0 and S_1 respectively. The second integrals on both sides of this equation have the value zero since G and F are harmonic at P_0 and P_1 respectively. The values of the first integrals are independent of the surfaces over which we integrate provided merely that we integrate over small closed surfaces surrounding the points P_1 and P_0 respectively. By taking the first of these surfaces as a small sphere with centre at P_1 we find at once

$$(6) \quad \int_{S_1} \frac{\partial G}{\partial n} dS = 4\pi.$$

If, then, we write

$$(7) \quad \int_{S_0} \frac{\partial F}{\partial n} dS = a,$$

we have

$$(8) \quad 4\pi F_{S_1} = a G_{S_0}.$$

Now let the quantities c_0 and c_1 become infinite, so that the sur-

faces S_0 and S_1 shrink down towards the points P_0 and P_1 respectively. We get as the limit of the last equation

$$(9) \quad \begin{aligned} F(x_1, y_1, z_1) &= \frac{a}{4\pi} G(x_0, y_0, z_0) = \frac{a}{4\pi} G(x_0, y_0, z_0; x_1, y_1, z_1) \\ &= \frac{a}{4\pi} G(x_1, y_1, z_1; x_0, y_0, z_0).^* \end{aligned}$$

In this formula we now regard the point P_0 as fixed, but P_1 as variable. It is important to notice that a is independent of x_1, y_1, z_1 , as is obvious from its definition (7). Accordingly (9) in combination with (3) and the well known formula

$$(10) \quad \begin{aligned} G(x_1, y_1, z_1; x_0, y_0, z_0) &= [(x_1 - x_0)^2 + (y_1 - y_0)^2 \\ &\quad + (z_1 - z_0)^2]^{-\frac{1}{2}} + g, \end{aligned}$$

where g is a harmonic function of x_1, y_1, z_1 at P_0 , gives us precisely the formula (2) for u which we wished to establish.†

Let us now consider the behavior of a harmonic function at infinity. We will assume $n > 2$.

Let

$$r^2 = x_1^2 + \dots + x_n^2$$

* We make use here of a fundamental theorem concerning Green's functions, the ordinary proof of which is merely the special case of the work just done in which we take for the function F the Green's function $G(x, y, z; x_0, y_0, z_0)$. Cf. Green's original Essay, § 6.

† The proof here given may easily be so modified as to avoid the use of Green's functions. For this purpose we apply Green's theorem to the two functions u and $1/r_1$, where r_1 is the distance from (x_1, y_1, z_1) to (x, y, z) . We use the same region as above except that S_1 is now a small sphere having its centre at P_1 ; and get in place of (4)

$$\begin{aligned} \int_{S_1} \left(u \frac{\partial(1/r_1)}{\partial n} - \frac{1}{r_1} \frac{\partial u}{\partial n} \right) dS &= \int_{S_0} \left(\frac{1}{r_1} \frac{\partial u}{\partial n} - u \frac{\partial(1/r_1)}{\partial n} \right) dS \\ &\quad + \int_S \left(\frac{1}{r_1} \frac{\partial u}{\partial n} - u \frac{\partial(1/r_1)}{\partial n} \right) dS, \end{aligned}$$

from which follows, if we let $\int_{S_1} \frac{\partial u}{\partial n} dS = a$,

$$\begin{aligned} 4\pi u(x_1, y_1, z_1) &= a[(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]^{-\frac{1}{2}} \\ &\quad + \int_S \left(\frac{1}{r_1} \frac{\partial u}{\partial n} - u \frac{\partial(1/r_1)}{\partial n} \right) dS. \end{aligned}$$

The integral which here remains is easily shown to be a harmonic function of (x_1, y_1, z_1) throughout S .

and suppose that $u(x_1, \dots, x_n)$ is harmonic when $r > R$, where R is some positive constant. It follows from a theorem due to Lord Kelvin * that if we let

$$x'_i = \frac{x_i}{r^2}, \quad r' = \frac{1}{r},$$

then

$$(11) \quad \frac{u(x_1, \dots, x_n)}{r'^{n-2}} = \bar{u}(x'_1, \dots, x'_n)$$

is a harmonic function of $x'_1, \dots, x'_n$ at all points for which $r' < 1/R$ except at the point $x'_1 = \dots = x'_n = 0$.

Let us now consider the different ways in which u may behave as the point $P = (x_1, \dots, x_n)$ moves off to infinity. It may become positively infinite for some ways in which P goes to infinity; negatively infinite for others. Let us, however, suppose that this is not the case. Then there exists a constant a such that $u + a$ does not vanish at any point for which $r > R$. Accordingly by (11) the function

$$\frac{u(x_1, \dots, x_n) + a}{r'^{n-2}}$$

is a harmonic function of $x'_1, \dots, x'_n$ which becomes infinite for every method of approaching the point $r' = 0$, and which can, therefore, by theorem II, be expressed in the form

$$\frac{b}{r'^{n-2}} + v(x'_1, \dots, x'_n)$$

where v is harmonic at the point $r' = 0$. We thus get

$$(12) \quad u(x_1, \dots, x_n) = (b - a) + r'^{n-2} v(x'_1, \dots, x'_n).$$

Accordingly, no matter how P moves off to infinity, u approaches the finite limit $b - a$. Hence the theorem

III. *The function u being harmonic when $r > R$, it either becomes both positively and negatively infinite for different ways of going to infinity, or it approaches one and the same finite limit for every method by which the point P recedes to infinity.*†

* Liouville's Journal, vol. 12 (1847), p. 259.

† We note in passing the following consequence of this theorem:

If for all values of $x_1, \dots, x_n$ the function u is harmonic and $u < M$ (or if $u > m$) then u is a constant.

If in particular u vanishes at infinity, we see from (12), in which now $b = a$, that

$$(13) \quad r^{n-2}u(x_1, \dots, x_n)$$

approaches a finite limit at infinity. Furthermore, by differentiating (12) with regard to r we see that

$$(14) \quad r^{n-1} \frac{\partial u}{\partial r}$$

approaches a finite limit at infinity. This behavior of the functions (13) and (14) is often postulated as an additional restriction on the function u , whereas we see that it follows as a consequence of the mere requirement that u vanish at infinity.

Let us now leave the consideration of the point at infinity and look again at a finite isolated singular point P_0 .

IV. *If a harmonic function u of two or more variables becomes infinite for no method of approaching an isolated singular point P_0 , then by suitably changing the definition of the function at this point it can be made harmonic there as well as elsewhere.*

In the case of two independent variables this theorem is suggested at once by a similar theorem for functions of a complex variable, and can be readily proved by the method suggested in the second footnote on page 455. I owe to a remark of Professor Osgood the following method by which the theorem can in all cases be established as an immediate consequence of theorems I and II.

For this purpose let us add to u the function

$$\left[\sum_1^n (x_i - \bar{x}_i)^2 \right]^{\frac{2-n}{2}}$$

or in the case $n = 2$ the function

$$\log \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

We thus have a harmonic function having P_0 as an isolated singular point and becoming infinite for every method of ap-

For in this case by III u approaches a finite limit c at infinity. Let P_0 be any point and describe a sphere with P_0 as centre. The average value of u on this sphere is known to be equal to the value of u at P_0 . But by taking the sphere large enough, the average value of u on the sphere can be made to differ from c by as little as we please. Therefore the value of u at P_0 must be precisely c . But P_0 was any point. Therefore u has the value c everywhere.

proaching this point. It therefore has the form (2), or if $n = 2$ the form (1). Accordingly we have

$$u = (C - 1) [\Sigma (x_k - \bar{x}_k)^2]^{\frac{2-n}{2}} + v \quad (n > 2)$$

$$u = (C - 1) \log \sqrt{(x - x_0)^2 + (y - y_0)^2} + v \quad (n = 2)$$

But since u does not become infinite as we approach P_0 , we have $C = 1$ and therefore $u = v$. Accordingly u is harmonic at P_0 .

I will add that precisely the same method may be applied to prove the following theorem:

V. *If a harmonic function u of two or more variables becomes infinite for some but not for all methods of approaching an isolated singular point P_0 , then it can be made to become both positively and negatively infinite (and therefore also to take on every real value an infinite number of times) by suitably approaching P_0 .*

2. The Elliptic Equation with two Independent Variables.

We will consider in this section the differential equation

$$(15) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = 0,$$

where a , b , and c denote real functions of the two independent real variables x , y analytic at every point of a region T of the x , y plane. We shall find it convenient to speak of a function u as being harmonic with regard to (15) at a given point, if at this point and throughout its neighborhood it has continuous first and second partial derivatives and satisfies (15). We will say that u is harmonic throughout a region with regard to (15) if it is harmonic at every point of this region with regard to (15). According to a remarkable theorem of Picard,* a function harmonic with regard to (15) at a point is analytic at that point.

In a recent dissertation † E. R. Hedrick has confirmed and made more precise a guess of Sommerfeld ‡ by proving the following theorem:

* *Journal de l'École Polytechnique*, Cah. 60 (1890).

† "Ueber den analytischen Character der Lösungen von Differentialgleichungen," Göttingen, 1901.

‡ *Encyclopädie*, II A 7 c, pp. 515 and 570.

If $P_0 = (x_0, y_0)$ is any point of T , there exists a function harmonic with regard to (15) throughout the neighborhood of P_0 , except at P_0 itself, and having the form

$$(16) \quad w = \log \sqrt{(x - x_0)^2 + (y - y_0)^2} \cdot U(x, y) + V(x, y)$$

where U is harmonic with regard to (15) at P_0 , $U(x_0, y_0) = 1$, and V is analytic at P_0 .

Now the theorem we wish to prove is this :

VI. If the function u harmonic with regard to (15) has an isolated singular point at P_0 * and if u becomes infinite for every method of approaching P_0 , then u has the form

$$(17) \quad u = Cw + v$$

where C is a constant, w is the function (16), and v is a function harmonic with regard to (15) at P_0 .

To prove this theorem let us begin by considering the special case of (15) in which $c \equiv 0$. Draw a circle S about P_0 as centre and so small that not only the function w (formula (16)) is harmonic with regard to (15) within and upon S , except at P_0 , but that a Green's function† $G(x, y; x_1, y_1)$ exists for every point $P_1 = (x_1, y_1)$ within S . Let $\bar{u}$ be the function which takes on the same values as u on S and is harmonic with regard to (15) throughout the interior of S , and form the function

$$F(x, y) = u - \bar{u}.$$

Let us now surround the points P_0 and P_1 (these points being supposed distinct) by small closed curves S_0 and S_1 respectively on which the functions F and G have the numerically large values c_0 and c_1 respectively; and let us suppose these values to be taken so large that S_0 and S_1 lie wholly outside of each other. Now apply the Riemann-Darboux extension of

* It must be clearly understood that we assume a, b, c to be analytic at P_0 . That is we consider not the fixed isolated singular points of the differential equation, but the movable singular points of its solutions.

† That is a function of (x, y) which vanishes on S , becomes logarithmically infinite at P_1 , and is harmonic with regard to the equation adjoint to (15) within and upon S except at P_1 . Cf. Sommerfeld, *Encyclopaedie*, Vol. II, p. 570, where it is stated that the existence of this Green's function follows from Mr. Hedrick's theorem. This is in fact the case, but only after Mr. Hedrick's work has been completed by an investigation of the dependence of w (formula (16)) on the parameters (x_0, y_0) ; an investigation which is also necessary for the purposes which Mr. Hedrick had in view.

Green's theorem (cf. Encyclopædie, Volume II, page 513), to the region bounded by S, S_0, S_1 . When we remember that throughout this region F satisfies (15) and G satisfies the adjoint of (15), while both of these functions vanish on S , we see that Green's theorem reduces to

$$\begin{aligned} \int_{S_1} \left[F \frac{\partial G}{\partial n} - G \frac{\partial F}{\partial n} - (a \cos \phi + b \sin \phi) FG \right] ds \\ = \int_{S_0} \left[G \frac{\partial F}{\partial n} - F \frac{\partial G}{\partial n} + (a \cos \phi + b \sin \phi) FG \right] ds, \end{aligned}$$

where n denotes the normal to the curves S_0 and S_1 which points away from the points P_0 and P_1 , and ϕ denotes the angle which this normal makes with the axis of x . Using formula (9) on page 515 of Sommerfeld's article* (Encyclopædie, Volume II) we see that the first member of this equation has the value $2\pi F(x_1, y_1)$. We have then

$$\begin{aligned} 2\pi F(x_1, y_1) = \int_{S_0} G \frac{\partial F}{\partial n} ds \\ + c_0 \int_{S_0} \left[-\frac{\partial G}{\partial n} + (a \cos \phi + b \sin \phi) G \right] ds. \end{aligned}$$

The last integral here is zero as we see at once by applying Green's theorem to the two functions G and 1, and to the region bounded by S_0 .† Applying the law of the mean to the first integral, as we have a right to do since $\partial F/\partial n$ does not change sign on S_0 , we can write

$$2\pi F(x_1, y_1) = G_{S_0} \int_{S_0} \frac{\partial F}{\partial n} ds.$$

Let us now allow the numerical value of c_0 to increase indefinitely, so that S_0 shrinks down toward the point P_0 . The quantity G_{S_0} then approaches $G(x_0, y_0; x_1, y_1)$ as its limit; and, since this is different from zero,‡ we see that the integral in

* There is a mistake of sign in this formula, as is easily seen by comparing it with the familiar formula for Laplace's equation which is a special case of it.

† Our restriction up to this point to the case $c = 0$ was in order to make 1 a solution of (15), this being essential for the step here taken.

‡ The equation (15) when $c = 0$ shares with the special case of Laplace's equation the property that a function harmonic with regard to (15) at a certain point cannot have a maximum or a minimum there, for otherwise by

the last formula also approaches a finite limit a . Thus we have

$$F(x_1, y_1) = \frac{a}{2\pi} G(x_0, y_0; x_1, y_1) = \frac{a}{2\pi} H(x_1, y_1; x_0, y_0)$$

where H denotes the Green's function for the adjoint of (15) (cf. Sommerfeld, p. 516). But H has the form (16) and therefore F , and so also u , have the form (17), as was to be proved.

Having thus proved Theorem VI in the special case $c \equiv 0$, we get the proof in the general case by means of the following transformation.

Let $\psi(x, y)$ be a function harmonic with regard to (15) at P_0 and having a value different from zero at this point. Then, making the transformation :

$$u = \psi t,$$

we find that t satisfies an equation of the form

$$(18) \quad \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + a_1 \frac{\partial t}{\partial x} + b_1 \frac{\partial t}{\partial y} = 0,$$

where a_1, b_1 are analytic at P_0 . Now if u becomes infinite for every method of approaching P_0 , t will do so too, and therefore, by what we have just proved, t becomes logarithmically infinite at P_0 . From this we readily infer that u has the form (17).

A second theorem concerning equation (15) is the following :

VII. *If a function harmonic with regard to (15) becomes infinite for no method of approaching an isolated singular point, then, by suitably changing the definition of the function at this point, it can be made harmonic there as well as elsewhere.*

The proof of this theorem being practically the same as that of theorem IV need not be repeated.*

A theorem analogous to V may also be proved.

subtracting a suitable constant, we should have a solution of (15) vanishing around a contour, which can be made as small as we please, and yet not vanishing identically, and this is well known to be impossible. Now $G(x_0, y_0; x, y)$, when regarded as a function of its last two arguments is, harmonic with regard to (15), and, since it vanishes on S and becomes negatively infinite at P_0 , it could not vanish within S unless it had a maximum at some point within S .

* We note in passing that theorems VI, VII admit of immediate extension first to the case in which the second member of (15) is a given analytic function of (x, y) , and second, by a change of independent variables, to the general linear differential equation with two independent variables and of the elliptic type. The statement of VII would require no modification in this general case, while the modification necessary in the statement of VI will be readily seen.

3. *More General Differential Equations.*

There seems little doubt that the results of this paper can be extended to all differential equations of the form

$$(18) \quad \sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{i=n} a_i \frac{\partial u}{\partial x_i} + cu = f \quad (n > 2)$$

where $a_1, \dots, a_n, c, f$ are analytic functions of the n independent variables $(x_1, \dots, x_n)$. The only difficulty here consists in the proof of the existence of a Green's function for a sufficiently small region S , which for the sake of simplicity we may take as spherical; i. e., a function which, except at an arbitrarily given point P_1 in S , is continuous and has continuous first and second partial derivatives throughout S , and satisfies the adjoint equation

$$(19) \quad \sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial x_i^2} - \sum_{i=1}^{i=n} \frac{\partial(a_i u)}{\partial x_i} + cu = 0,$$

and vanishes on S , while at P_1 it has the form

$$(20) \quad U \cdot r^{2-n} + V,$$

where r denotes the distance from P_1 , while U and V are analytic at P_1 , and U has the value 1 at this point, and satisfies (19).

Granting the existence of this Green's function, the methods already used give at once the following theorem:

VIII. *If throughout the neighborhood of a point P_0 at which the coefficients of (18) are analytic, a function u is continuous and has continuous first and second partial derivatives and satisfies (18), but if all these conditions are not satisfied at P_0 , then*

(a) *if u becomes infinite for every method of approaching P_0 it has the form*

$$u = U \cdot r^{2-n} + V$$

where r is the distance from P_0 , U and V are analytic at P_0 , and U satisfies the equation obtained from (18) by replacing the second member by zero;

(b) *if u becomes infinite for no method of approaching P_0 it can by a suitable change of definition at this point be made continuous there, in which case it will also have continuous first and second derivatives at P_0 and will satisfy (18) there;*

(c) *if u becomes infinite for some but not for all methods of approaching P_0 , it takes on every value an infinite number of times in the neighborhood of P_0 .*

HARVARD UNIVERSITY, CAMBRIDGE, MASS.,
January, 1903.

ERRATA IN GAUSS'S "TAFEL DER ANZAHL DER CLASSEN BINÄRER QUADRATISCHER FORMEN."

BY THE LATE PROFESSOR A. M. NASH.

(Communicated by Professor E. B. ELLIOTT.)

ON page 141 of the present volume 9 of the *BULLETIN* Professor Hewes has exhibited the details, on Cayley's plan, of the classes of properly primitive reduced forms with the two negative determinants — 468 and — 931, which Perott has shown must be added to those given as irregular by Gauss. There is at present in my charge a mass of calculation and tabulation prepared by the late Professor A. M. Nash, of the Indian Educational Department, who died in 1895 while still engaged on the work. He seems to have possessed methods by which he could tabulate irregular determinants with few if any probable omissions. Unfortunately however his numerical work is accompanied by but little explanation of the methods and their justification. The determinants which he has tabulated as irregular cover a range up to about — 20,000, even indices, indices 3 and 9, and index 5 being separately dealt with. For about half of them the details of classes are worked out on Cayley's plan, and the composition of the principal genus given. He has also worked out in detail the classes of binary quadratic forms with all negative determinants, both regular and irregular, up to — 2,600, the results in this section of his research being marked nearly to the end "checked Gauss."

To print the whole is not feasible. The quantity alone would be prohibitive even had the principles and conclusions been left in form for publication. It is hoped however that the manuscript may be placed in some library to which mathematicians who may be desirous in the future of testing and extending the tables, or may be engaged in research on the subject, can have access. What Professor Hewes's note now suggests to me is the desirability of printing the following transcript of a list which is headed by Professor Nash "Errata in Table of Negative Determinants, Gauss, Vol. II." The list

includes many corrections not given by Schering in his appendix to Gauss's volume, or by Perott.

	<i>For.</i>	<i>Read.</i>		<i>For.</i>	<i>Read.</i>
468	<i>Reg.</i>	<i>Irr.</i> 2	2331	<i>Reg.</i>	<i>Irr.</i> 2
485	IV 4	IV 5	2196	<i>Reg.</i>	<i>Irr.</i> 2
544	<i>Reg.</i>	<i>Irr.</i> 2	2180	<i>Reg.</i>	<i>Irr.</i> 2
547	<i>Reg.</i>	<i>Irr.</i> 3	2304	<i>Reg.</i>	<i>Irr.</i> 2
557	II 11	II 13	2320	<i>Reg.</i>	<i>Irr.</i> 2
647	I 25	I 23	2624	* 3 *	* 2 *
894	IV 6	IV 7	2336	<i>Reg.</i>	<i>Irr.</i> 2
931	<i>Reg.</i>	<i>Irr.</i> 3	2900	<i>Reg.</i>	<i>Irr.</i> 2
933	IV 3	IV 4	2188	<i>Reg.</i>	<i>Irr.</i> 3
972	<i>Reg.</i>	<i>Irr.</i> 3	2085	VIII 5	VIII 4
993	IV 4	IV 3	2096 in IV 6	2096	2097
1116	IV 9	IV 6	2204	IV 11	IV 13
1261	II 10	IV 5	2221 in IV 9	2221	2224
1367	I 27	I 25	2376 in IV 12	2376	2366
1396	IV 7	II 14	2448	<i>Irr.</i> 2	<i>Reg.</i>
1508	<i>Reg.</i>	<i>Irr.</i> 2	6032	<i>Reg.</i>	<i>Irr.</i> 2
1598	<i>Reg.</i>	<i>Irr.</i> 2	6068	<i>Reg.</i>	<i>Irr.</i> 2
1660	IV 4	Omit	6084	<i>Reg.</i>	<i>Irr.</i> 2
1683	II 9	II 6	6148	<i>Reg.</i>	<i>Irr.</i> 2
1700	—	IV 12	6176	<i>Reg.</i>	<i>Irr.</i> 2
1701	<i>Reg.</i>	<i>Irr.</i> 3	9104	<i>Irr.</i> 2	<i>Reg.</i>
1725	<i>Reg.</i>	<i>Irr.</i> 2	9108	<i>Reg.</i>	<i>Irr.</i> 2
1796	IV 10	II 20	9156	<i>Reg.</i>	<i>Irr.</i> 2
1836	<i>Reg.</i>	<i>Irr.</i> 3	9216	<i>Reg.</i>	<i>Irr.</i> 2
1872	<i>Reg.</i>	<i>Irr.</i> 2	9324	<i>Reg.</i>	<i>Irr.</i> 2
1937	—	IV 12	9513	<i>Irr.</i> 2	<i>Reg.</i>
1982	IV 12	Omit	9554	<i>Reg.</i>	<i>Irr.</i> 2
1940	IV 8	IV 10	6075	<i>Irr.</i> 3	<i>Irr.</i> 9

THE LOGARITHM AS A DIRECT FUNCTION.

BY DR. EMORY MCCLINTOCK.

(Read before the American Mathematical Society, February 28, 1903.)

IN a paper of the same title published in the *Annals of Mathematics* for January, 1903, Mr. J. W. Bradshaw defines $\log x$ as a direct function of x , namely,

$$\log x = \int_1^x x^{-1} dx.$$

Attention being thus drawn to the subject, I think the time opportune to repeat and amplify a proposition of my own for the same general purpose.

In 1879 (*American Journal of Mathematics*, II, 101, etc.) I spoke of "the difficulty of comprehending logarithms," and quoted De Morgan's dictum that "the only definition of $\log x$ used in analysis is y , where $e^y = x$." After discussing this definition I said, "Another and, when duly weighed, most satisfactory definition may be derived from any one of an unlimited number of vanishing fractions, special cases of the general form $\log x = h^{-1}(x^{(1-a)h} - x^{-ah})$, where h is infinitely reduced. * * * This fraction is doubtless novel, though one case of it, where $a = 0$, is known. Even that case has not, I presume, been suggested heretofore as a definition. * * * The various theorems pertaining to logarithms may be derived with the utmost facility by the aid of these vanishing-fraction definitions. Thus, if $a = 0$, we have by expansion

$$\log(1+x) = \frac{(1+x)^h - 1}{h} [h=0] = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots"$$

To develop this proposition more fully, let us consider the function $y = h^{-1}(x^h - 1)$. Let k be positive, and, first, let $h = k$. When $x = 1$, $y = 0$; when $x = \infty$, $y = \infty$; and as x increases continuously from 1 towards ∞ , there is one and only one corresponding value of y , which increases accordingly from 0 towards ∞ . Secondly, let $h = -k$. Here again, when $x = 1$, $y = 0$; and, in the function $y = k^{-1}(1 - x^{-k})$, as x increases continuously from 1 towards ∞ , there is one and only one corresponding value of y , which increases accordingly from 0 to k^{-1} , a limit which tends towards ∞ if k tends towards 0. When $h = -k$, $y = k^{-1}(x^k - 1)x^{-k}$, which differs only by the factor x^{-k} from the value of y when $h = k$. The smaller k is taken, the nearer this factor is to 1, so that the limit of the value of y , for $h = 0$, is the same whether h is positive or negative, while

$$y_{[h<0]} < \lim_{h \rightarrow 0} y < y_{[h>0]}.$$

The limit is therefore a 1 to 1 function of $x > 1$. When $x = 1$, the limit is 0. When $0 < x < 1$, let $x = u^{-1}$, where $u > 1$; then we have $h^{-1}(x^h - 1) = h^{-1}(1 - u^h)x^h$, and, since $\lim_{h \rightarrow 0} x^h = 1$, $\lim_{h \rightarrow 0} h^{-1}(x^h - 1) = -\lim_{h \rightarrow 0} h^{-1}(u^h - 1)$.

Let us define the logarithm of x (positive) as $\lim_{h \rightarrow 0} h^{-1}(x^h - 1)$, and denote it by $\log x$. We have just found that $\log(x^{-1}) = -\log x$. Since $\lim_{h \rightarrow 0} b^h = 1$,

$$\log a = \lim_{h \rightarrow 0} h^{-1}(a^h b^h - b^h) = \log(ab) - \log b.$$

This is the chief property of logarithms. Hence, $\log(a^2) = 2 \log a$, $\log(a^n) = n \log a$, and if $b = a^n$, $\log(b^{1/n}) = 1/n \log b$, which might be used, as by Mr. Bradshaw from another definition of $\log x$, to show that for every positive number b there exists one and only one positive n th root. Here n is a whole number. It follows that $\log b^{m/n} = m \log(b^{1/n}) = m/n \log b$. If we take n incommensurable, let $a = b^n$, where m is an integer, $b = 1 + c$, and $-1 < c < 1$. Employing the binomial expansion,

$$\begin{aligned} \log(b^n) &= \lim_{h \rightarrow 0} h^{-1}[(1 + c)^{nh} - 1] = n(c - \frac{1}{2}c^2 + \frac{1}{3}c^3 - \dots) \\ &= n \log(1 + c) = n \log b. \end{aligned}$$

Hence

$$\log(a^n) = \log(b^{mn}) = m \log(b^n) = nm \log b = n \log a.$$

That the continuous function $\log x$ has a continuous derivative x^{-1} may be shown thus, with $\Delta x < x$:

$$\frac{d \log x}{dx} = \lim_{\Delta x \rightarrow 0} \lim_{h \rightarrow 0} h^{-1}(\Delta x)^{-1}[(x + \Delta x)^h - x^h].$$

If we expand the part within the brackets and divide the resulting series throughout by $h\Delta x$, we have

$$\begin{aligned} \frac{d \log x}{dx} &= \lim_{\Delta x \rightarrow 0} \lim_{h \rightarrow 0} \left[x^{h-1} + \frac{1}{2}(h-1)(\Delta x)x^{h-2} + \right. \\ &\quad \left. \frac{1}{2 \cdot 3}(h-1)(h-2)(\Delta x)^2 x^{h-3} + \dots \right]. \end{aligned}$$

If we first put $h = 0$, the part within brackets becomes $x^{-1} - \frac{1}{2}(\Delta x)x^{-2} + \frac{1}{3}(\Delta x)^2 x^{-3} - \dots$, which is x^{-1} when $\Delta x = 0$. If we first put $\Delta x = 0$, the part within brackets becomes x^{h-1} , which is x^{-1} when $h = 0$.

THE THEORY OF AUTOMORPHIC FUNCTIONS.

Vorlesungen über die Theorie der Automorphen Functionen.
 Von ROBERT FRICKE und FELIX KLEIN. Leipzig, B. G. Teubner. Bd. I: *Die gruppentheoretischen Grundlagen*, 1897, xiv + 634 pp. Bd. II: *Die functionentheoretischen Ausführungen und die Anwendungen*; Erste Lieferung: *Engere Theorie der Automorphen Functionen*, 1901, 282 pp.

SIDE by side with the growth of general function theory special classes of functions have developed in which the general theories have found abundant opportunity to display their fertility and power. On the other hand, the study of special functions has repeatedly afforded the stimulus and suggested the path for new investigations along general lines. When, in addition, the intrinsic value and usefulness of such functions as the elliptic, hyperelliptic, abelian, hypergeometric, Bessel, etc., is taken into consideration, it is readily seen that the cultivation of such special fields is scarcely second in interest and importance to that of the general theory itself.

Among the various classes of special functions which have hitherto engaged the attention of mathematicians, that of most recent origin, and of by far the largest content (at least potentially) is the automorphic functions. This vast subject, the growth of the past quarter of a century, owes its rapid development to the genius and assiduity of the two eminent mathematicians, Klein and Poincaré, as well as to the comparatively high state of perfection of other mathematical disciplines which have been forced to contribute their assistance to this new field. Klein and Poincaré have each brought to this subject a breadth of knowledge and a corresponding wealth of ideas truly remarkable at the present day when multiplicity of interests hardly permits the investigator any other choice than to specialize within limits more or less narrow.

Klein in particular has by precept and example urged the importance of a closer unity among all departments of mathematical thought, and the great advantage to be derived from bringing to the aid of any one field the combined resources of all the others. It is in his work on the automorphic functions that he has given the most brilliant illustration of this mode of

treatment. In this work he has been ably seconded by the efforts of the pupils and co-laborers he has been so fortunate as to gather about him at the Göttingen school, among whom Robert Fricke is especially deserving of the highest praise for the ability and industry which he has brought to the arduous labor of preparing for publication the volumes now before us, and for the wealth of material which he himself has contributed to the subject.

These volumes form the sequel to, and final elaboration of the ideas contained in the volumes previously published, namely, the *Ikosaeder* (by Klein), and the *Elliptische Modulfunctionen** (by Klein and Fricke). The former of these works deals exhaustively with the finite groups of transformations on a single variable, and the functions associated with them. The two large volumes of the latter treat in a very elaborate manner the modular group, and the general ideas necessary to be followed out in the study of all such groups and their functions. One of the principal objects of this fullness of discussion in the *Modulfunctionen* is to pave the way for a subsequent discussion of the general theory of automorphic functions without the restraint of burdensome details.

The advantage of thus disposing of preliminary details, and familiarizing the reader with many of the fundamental ideas as applied to a concrete example is clearly perceived when we observe how completely are the various elements in the present work fused into an organic whole, each part in strong and vital contact with every other part. The degree of abstractness and concentration thus attained is necessary for a forceful treatment of a subject which seeks to embrace in one view such an infinite variety and complexity of phenomena.

The theory of automorphic functions has for its object to investigate the discontinuous groups of linear transformations of a single variable ζ , and to study the properties of functions which are invariant for the transformations of any given group.

The most important aid in this work comes from geometry,† which affords a concrete representation of the values of the

* The interesting review of the *Modulfunctionen* by Professor Cole, and the general survey of Klein's work and ideas which it contains, renders it unnecessary for us to give more than a passing notice to the antecedents of the present work. See BULLETIN, 1st Series, vol. 1 (1892), p. 105.

† Geometry is, in fact, at the present stage of development of the subject, an indispensable tool. To the lack of geometrical aids is in no small measure to be attributed the meager success that has attended the various attempts to investigate automorphic functions of more than one variable.

complex variable ζ and pictures the effect of a group of transformations applied to this variable. In the Automorphe Functionen an introductory chapter of about sixty pages is accordingly devoted to the discussion of some geometrical notions not explained in the Modulfunktionen.

The first geometrical interpretation of the complex variable ζ is serviceable only for a restricted, but very extensive class of groups, the *fundamental circle groups* (Hauptkreisgruppen). The values of ζ are associated with the homogeneous coördinates (z_1, z_2, z_3) of points in a plane by means of the relation

$$= \zeta \frac{z_2 + \sqrt{z_2^2 - z_1 z_3}}{z_3}.$$

Collineations which leave the conic

$$(1) \quad z_2^2 - z_1 z_3 = 0$$

unaltered correspond to linear transformations

$$(2) \quad \zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}$$

on ζ . Taking the conic (1) as the absolute for a system of non-euclidean measurement of distance, the transformations (2) leave distances unaltered, and hence they are called (in a generalized sense) motions. Congruent figures, that is, figures which transform into each other have like (non-euclidean) areas. The interior of the conic, regarded for convenience as an ellipse, is called the "hyperbolic" plane.*

Out of the ∞^6 transformations of the continuous group (2) let us suppose a discontinuous group Γ selected by any suitable definition. This group is then *properly*, or *improperly* discontinuous, according as the fundamental region for Γ has a finite or an infinitesimal area. By "fundamental region" is meant a division of the hyperbolic plane such that no two points of the region are congruent, while every point without the region is congruent to some point within. The properties

* The cases in which the conic is imaginary (giving the "elliptic" plane), or breaks up into the circular points at infinity (giving the ordinary, or "parabolic" plane) lead to groups of finite order, or groups associated with the elliptic functions. These cases receive their full share of attention in the work under review, but from lack of space we pass them by without discussion.

of a given group are all implicitly involved in the geometrical properties of the fundamental region which it defines, and, accordingly, a large portion of what follows in volume I is devoted to a geometric study of the fundamental region and the network of congruent regions which arise from it by transformations of the group.

Although, evidently, the fundamental region can be selected with a high degree of arbitrariness, it can always be defined (and this in an infinity of ways) so as to be bounded by straight lines. The fundamental region is then a polygon, and the group belonging to it is called a polygon group.

A particularly useful choice for a fundamental polygon, and one which plays a leading rôle throughout the book, is determined as follows: Let $C_0, C_1, C_2, \dots$ be any set of congruent points. Around the points C_i construct circles K_i of equal radii r , meaning by circle in this connection the locus of all points whose non-euclidean distance from a given C_i is constant. Assuming r at first sufficiently small so that no two circles collide, let the radii increase simultaneously and at the same rate for all the circles K_i with the understanding that the lengthening of any particular ray emanating from C_i shall cease the instant it meets a like ray emanating from a neighboring point C_j . Imagine this process to continue until the entire hyperbolic plane is filled without gap and without overlapping. The locus of the end points of the rays departing from a center C_i will consist of a closed chain of straight lines forming the boundary of a convex polygon. This region, called the *normal polygon*, is evidently a fundamental region for the group.*

Now let the values of ζ be associated in the ordinary way with the points of a complex plane. Then, assuming $z_2^2 - z_1 z_3$ to be negative for the interior of the ellipse, to each point of the hyperbolic plane correspond two points of the ζ -plane representing conjugate imaginary values of ζ . To each point on the ellipse corresponds one point on the real ζ -axis, and to

* Each of the points C_i is the center of a normal polygon. This polygon clearly contains within itself all points which are nearer to C_i (in the non-euclidean sense) than to any congruent point. Each polygon is equal in all respects to every congruent polygon (still speaking in the non-euclidean sense). This picturing of the group with a network of equal polygons filling up the hyperbolic plane, is seen to be a generalization of the idea of a network of equal parallelograms filling the ordinary (or parabolic) plane, with which the theory of elliptic functions has made us familiar.

each point outside the ellipse corresponds a pair of points on the real axis. On account of the one-to-two correspondence it is evidently sufficient to take into consideration only half of the ζ -plane. Moreover, by a linear transformation of ζ the real axis changes into a circle C the interior of which has a one-to-one correspondence to the half ζ -plane and hence to the hyperbolic plane.

In the hyperbolic plane the polygons crowd together in infinite number along the ellipse which forms its boundary, since this is the infinite element in the plane. Any set of congruent points, such as the C_i above mentioned, will have limiting points, or points of accumulation, on this boundary. These limit points may cover the ellipse everywhere densely, in which case the ellipse, and correspondingly the circumference of the circle C , forms a natural boundary for the group and its network of polygons.* The group is then called a *limit circle group* (Grenzkreisgruppe). On the other hand, segments of the ellipse (or circle C) of finite length may be free from limiting points. These two classes, which include all groups whose fundamental regions fill the hyperbolic plane, are specified by the general term, *fundamental circle groups*.

The foregoing method of associating the ζ values with the real points of the hyperbolic plane is of practical value only when the substitutions of the group have real coefficients since it is only in that case that real points in the hyperbolic plane are transformed into real points.* This difficulty is obviated by a more general procedure. A quadric surface is taken in ordinary space, for convenience the sphere

$$(3) \quad z_1^2 + z_2^2 + z_3^2 - z_4^2 = 0.$$

The values of ζ are associated with points on the surface of the sphere in the familiar manner of the Riemann function theory by means of the relation

* It follows that any function, invariant for transformations of the group, has the circle for natural boundary and cannot be continued over this boundary which is everywhere filled with essential singularities of the function.

* That there is a practical necessity for the restriction to real coefficients is not mentioned by our authors, although they may have expected it to be inferred. This inference, however, is not likely to be at once drawn by the reader, since the reason assigned in the text is that only in this case are groups of "motion," or motion combined with inversion, obtained, without explaining why it is desired to exclude from consideration all groups other than these.

$$(4) \quad \zeta = \frac{z_1 + iz_2}{z_4 - z_3}.$$

Every collineation which transforms the sphere into itself subjects ζ to a linear transformation of one of two kinds,

$$(5) \quad \zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \quad \bar{\zeta}' = \frac{\alpha\bar{\zeta} + \beta}{\gamma\bar{\zeta} + \delta},$$

when $\bar{\zeta}$ is the conjugate of ζ . Taking the surface of the sphere as the absolute for a system of non-euclidean measurement in space, the interior of the sphere is called the hyperbolic space. The non-euclidean distance between two points is unchanged by the transformations (5). Those of the first type are "movements" of the hyperbolic space, while one of the second type is a reflection, or symmetrical transformation with respect to a certain plane of symmetry.

The values of $\zeta = \xi + i\eta$ are likewise represented by the coördinates of points in a complex plane referred to ξ, η rectangular axes. A third rectangular coördinate ϑ in space is introduced. Then assuming

$$\vartheta = \pm \frac{\sqrt{z_4^2 - z_1^2 - z_2^2 - z_3^2}}{z_4 - z_3},$$

this equation combined with (4) establishes a correspondence between the ξ, η, ϑ space and the hyperbolic space such that to a point within the sphere correspond two points situated symmetrically with respect to the ζ plane. The portion of the ξ, η, ϑ space situated on one side of the ζ plane has a one-to-one correspondence with the hyperbolic space. This is called the ζ half-space. To each plane of the hyperbolic space corresponds a half sphere of the ζ half-space which is orthogonal to the ζ plane.

Consider now a discontinuous group Γ of transformations (5). A fundamental region for Γ will consist of a limited portion of the hyperbolic space (or of the ζ half-space) of largest possible extent subject to the condition that no two congruent points are within this region. Such a fundamental region can be selected in an infinity of ways so as to be bounded by planes alone. If this fundamental region be transformed by all the substitutions of the group, the totality of congruent regions

thus obtained will fill the hyperbolic space without gap and without overlapping. These polyhedra may or may not intersect the surface of the sphere. In the former case the surface is divided up into a network of polygons forming fundamental regions for the ζ transformations, which, by the mediation of the above correspondence (4), may be depicted on the ζ plane, while at the same time the network of polyhedra are represented by polyhedra in the ζ half-space whose faces are half spheres orthogonal to the ζ plane. The group is then properly discontinuous in the ζ plane (or on the ζ sphere).

If, in particular, the group consists of all the transformations which leave unchanged a point P outside the sphere, the polar plane of P and its circle of intersection with respect to the sphere is invariant for the group, which is accordingly a fundamental circle group. From this point of view it is called a *hyperbolic rotation group*. If the point P is on the sphere, we have the *parabolic rotation groups* with one limit point, to which the elliptic functions are related. Finally, when P is within the sphere, the polar plane does not intersect the surface in real points, and hence the group has no limit points. There are accordingly only a finite number of fundamental regions, and the group is finite. It is called an *elliptic rotation group*.

It is found that the use of the polyhedra in the hyperbolic space is an especial convenience and simplification in the study of properties of non-rotation groups, and indispensable for the investigation of groups which are improperly discontinuous on the surface of the sphere (or in the ζ plane). An example of the latter kind to which considerable space is allotted is the Picard group which consists of all the substitutions of ζ formed with complex integer coefficients whose determinant is 1 or i .

Returning to the polyhedra into which the hyperbolic space has been divided by the transformations of the group, it is further noticed that these can be determined in a manner exactly analogous to the determination of the normal polygons in the hyperbolic plane. A set of congruent points $C_0, C_1, \dots$ is selected at random and small spheres (in the non-euclidean sense) of equal radii r described about each. Then r is imagined to increase simultaneously for all these spheres, with the understanding that any ray emanating from C_i stops increasing when it meets a ray proceeding from a neighboring center. When this process is carried to an end, the hyperbolic space will be filled without gaps and without overlapping by poly-

hedra, all of whose faces are planes. A polyhedron determined in this way is called a *normal polyhedron*. The study of the normal fundamental polyhedra here plays the same rôle in investigating the properties of the group as the normal polygon does in the case of the fundamental circle groups.

Particular attention is devoted to those non-rotation groups whose normal polyhedra intersect the surface of the sphere in one or more distinct networks of polygons. A classification of groups is based on the nature of these nets and their limit points.

There may be a single net completely covering the sphere. The limit points are then isolated, and if there are more than two of them, the number is infinite. On the other hand, the number of nets may be two, or infinite. In either case the number of limit points is infinite since the nets are separated from each other by these points. Hence non-rotation groups may be classified into : * (a) those with two limit points ; (b) those with an infinity of limit points. Division *b* is classified still further according as the number of nets is (1) *one*, the limit points being isolated ; (2) *two*, the limit points forming a non-analytic curve separating the two nets ; (3) *infinite*, the infinity of nets being separated from each other by an infinity of limit curves formed by limit points of the group. This last case is further subdivided according as the limit curves are (at least in part †) non-analytic, or consist entirely of circles, and still further as the nets are simply or multiply (always with an infinite multiplicity) connected.

A kind of converse question naturally arises, namely, in how far can an arbitrarily chosen polygon serve as the fundamental region for a group of linear transformations, and thus serve to define and generate the group ? The restrictions under which a polygon can be chosen are determined for groups in the hyperbolic plane (fundamental circle groups). The like problem is then solved for polyhedra in the hyperbolic space and the results made useful for the determination of polygons suitable for defining non-rotation groups by means of the theorem : any polygon P_0 (on the surface of the sphere) is a fundamental

* These are divisions II and IV in the book. See pp. 164-5. Division I comprises the ordinary cyclic groups formed by the repetition of a single operation, and division III comprises the rotation groups already noticed above.

† If analytic curves occur, they must be circles.

region for some discontinuous group of transformations provided that a polyhedron can be constructed intersecting the sphere in P_0 and satisfying the restrictions which polyhedra are subject to in order to be usable for fundamental regions in the hyperbolic space.

A new kind of fundamental region, the *canonical polygon*, is next introduced which greatly facilitates the discussion of the generating operations of the group and the relations existing among them. To make clear the idea, it is observed that the sides of a fundamental polygon must be congruent in pairs, and the vertices in cycles. If the sides of a fundamental polygon be deformed so as to bring together and unite congruent lines and points, a closed surface F is obtained of a certain genus p which is called the genus of the group. The surface F may be conveniently thought of as an ordinary Riemann surface.

Consider now the reverse process. Take any Riemann surface and draw from an arbitrary point E a canonical system of cross-cuts which reduces the surface to a simply connected one. It may then be deformed into a polygon suitable for the fundamental region of a group. The opposite edges of the cross-cuts become congruent edges of the polygon, and the substitution which brings an edge into coincidence with a congruent edge is one of the generators of the group. If the fundamental polygon is assumed to have vertices which are fixed points for parabolic or elliptic substitutions, then a certain number of points $e_1, e_2, \dots, e_n$ are selected in the Riemann surface and cuts drawn from E to each of these points. If e_i is to be a fixed point of an elliptic substitution of finite period, then the deformation must be carried out so that in the end the angle at e_i is an aliquot part of π . If, on the other hand, e_i is to be a fixed point for a parabolic substitution, the angle must be zero after the deformation. The polygon so obtained is the canonical polygon. It has $2n + 8p$ edges which are congruent in pairs. It is possible to unite some of the congruent edges by a suitable deformation of the polygon. The $n + 4p$ substitutions which transform congruent edges into each other are generators of the group. Certain relations exist among these which readily permit their number to be reduced to $n + 2p$ among which a single relation exists (in addition to the relations of the form $V' = 1$ for the elliptic substitutions).

The second division of volume I is devoted to an application of the geometrical foundation principles, as laid down in the

first division, to the detailed study of polygon groups. The normal and canonical polygons are made the leading instruments of investigation, and the cases of the elliptic and parabolic rotation groups, the non-rotation groups with two limit points, the hyperbolic rotation groups, and the non-rotation groups with an infinity of limit points are considered in turn. The first three cases are comparatively simple and easily disposed of. The fourth case is treated at great length. The important principle is first deduced that for any group Γ the normal polygon with a given center C is the region common to the normal polygons (with the same center) of all the groups whose composition generates Γ . This principle is applied in particular to the cyclic subgroups of Γ .

Various properties of the sides and corners are next considered. It is shown for example that, while at an accidental corner (that is, one which is not a fixed point for a substitution of the group) in general only three polygons meet, for special positions of the center C more than three polygons meet in such a point. All polygons having the same genus p and number n of fixed vertices are said to be of the same kind and of character (p, n) . Polygons of the same kind having a like number of sides, the sides being congruent in the same order, are said to be of the same *type*. The type is *ordinary* or *special* according as the accidental corners occur in cycles of three and the fixed corners in cycles of one, or not. The relations which are found to exist between corners, sides, and number of cycles show that the occurrence of a special type involves a reduction of the number of sides s . Special types with $s - 4$ sides occur with each group and arise from a particular choice of the center C . Special types in which the number of sides is diminished by six or more below the number belonging to the ordinary type can occur only for special groups. There are called *singular groups*.†

* In fact C will lie either on the ellipse, or on a certain curve of the third degree.

† Our authors do not attempt to develop a theory of singular groups, but content themselves with remarking on the possible importance of these groups in relation to the theory of algebraic functions. The reader of the *Automorphe Functionen* will have frequent occasion, as here, to notice the occurrence of gaps in the development of the subject. While the work under review may rightly be regarded as presenting a systematic and well-developed theory of the automorphic functions, a vast amount of work yet remains to be done. To mention only one further instance, it may be remarked that the theory of non-rotation groups has not yet been systematized, and in spite

The second chapter in this division of the book deals with the canonical polygon with special reference to the form it may take in the hyperbolic plane, and the transformations it may undergo on account of transformation from one canonical system of cross-cuts in the Riemann surface to any other. The details are first worked out for polygons of character $(0, 3)$, $(1, 1)$, and $(0, n)$. The methods being made clear by these simplest cases, the case (p, n) is then treated in its generality. The remainder of the chapter is devoted to an extended discussion of the *moduli* (Moduln). The moduli are the parameters on which the substitution coefficients depend, and which are invariant when the group is transformed by any linear substitution. The groups so obtained are said to belong to the same class, and the moduli form a system of numbers characteristic for that class. As the groups of the same class have like structure, a determination of the properties of the class acquaints us with the properties of any group of the class. Any particular set of possible values for the moduli determines a class, and the number manifoldness of all the classes is measured by that of the moduli.

The third and last division of volume I treats of the arithmetic definition of discontinuous groups. This subject, although of fundamental importance, offers at the present time only fragmentary results on account of the great difficulties that it has to overcome.

The first chapter determines the arithmetic characteristics of the rotation subgroups of the Picard group. This is accomplished by the introduction of the Dirichlet and the Hermite quadratic forms, concerning which the following two theorems are proved: Every hyperbolic or loxodromic cyclic subgroup transforms into itself a particular Dirichlet form having for determinant a non-square; and, conversely, every such form is invariant for some such subgroup. Again, every fundamental circle group contained within the Picard group transforms into itself a particular indefinite Hermite form; and conversely, every such form is reproduced by some fundamental circle subgroup of the Picard group.

The second chapter discusses the groups that reproduce cer-

of the considerable space devoted to this subject and the powerful aid afforded by the introduction of the fundamental polyhedron in the hyperbolic space, this portion of the field can hardly be said to have been more than lightly touched upon. (Compare the remarks on page 441, Vol. I.)

tain ternary and quaternary forms. This includes a large and important class of groups, among which the Picard group occurs as the reproducing group of a particular quaternary form. A considerable number of illustrative cases are worked out.

The third chapter deals with groups whose substitutions have coefficients which are integers within a given field of algebraic numbers.

Volume II, of which the first part only has appeared, is devoted to a study of the automorphic functions belonging to the groups treated in the first volume. By an automorphic function of an independent variable ζ is meant a function $\phi(\zeta)$ which is unaltered when ζ is transformed by any substitution

$$(6) \quad \zeta' = \frac{\alpha_k \zeta + \beta_k}{\gamma_k \zeta + \delta_k}$$

of a given group Γ ; that is,

$$\phi\left(\frac{\alpha_k \zeta + \beta_k}{\gamma_k \zeta + \delta_k}\right) = \phi(\zeta).$$

It is further required that $\phi(\zeta)$ have no essential singularity at any point of the fundamental region of Γ , and that within this region it is uniform and without branching. Accordingly $\phi(\zeta)$ is expansible in the vicinity of an ordinary point ζ_0 in powers of $\zeta - \zeta_0$, and in powers of $1/\zeta$ in the vicinity of the infinite point.

If ζ_0 and ζ'_0 are the fixed points of an elliptic substitution of period l , then since the corresponding substitution can be written in the form

$$\frac{\zeta' - \zeta_0}{\zeta' - \zeta'_0} = e^{\frac{2\pi i}{l}} \frac{\zeta - \zeta_0}{\zeta - \zeta'_0},$$

the function $\phi(\zeta)$ can be expanded in the vicinity of ζ_0 in a series of ascending powers of

$$\left(\frac{\zeta - \zeta_0}{\zeta - \zeta'_0}\right)^l.$$

Every parabolic substitution can be written in the form

$$\frac{1}{\zeta' - \zeta_0} = \frac{1}{\zeta - \zeta_0} + \gamma,$$

in which ζ_0 is the fixed point. In the vicinity of this point $\phi(\zeta)$ can therefore be expanded in a series of ascending powers of $e^{\frac{2\pi i}{\gamma} \cdot \frac{1}{\zeta - \zeta_0}}$.

The first step in the development of the theory of automorphic functions is to prove that such functions exist for every group Γ which is properly discontinuous in the ζ -plane. This is accomplished by means of Riemann's general existence theorem, the details of the proof following the methods of Schwarz and Neumann.

If $\zeta = \xi + i\eta$, it is first shown that an automorphic potential $u(\xi, \eta)$ can be found which satisfies prescribed boundary conditions, and is uniform and everywhere continuous in the fundamental region of Γ except at one prescribed point ζ_0 where it becomes discontinuous like the real part of $1/(\zeta - \zeta_0)$.

The conjugate potential v is defined by the integral

$$v(\zeta, \eta) = \int_{(\xi_0, \eta_0)}^{(\xi, \eta)} \left(\frac{\partial u}{\partial \bar{\xi}} d\eta - \frac{\partial u}{\partial \eta} d\xi \right).$$

It is shown that dv has the automorphic character, and that hence v is reproduced by any substitution of the group with the addition of a constant. This constant is zero for groups of genus zero, and in that case the function

$$\phi(\zeta) = u(\xi, \eta) + iv(\xi, \eta)$$

is an automorphic function of ζ .

If $p > 0$, the expression $Z = u + iv$ behaves on the closed Riemann surface F , on which the fundamental region of Γ is depicted, like an elementary integral of the second kind. Hence, according to the well known Riemann method, if $Z_1, Z_2, \dots, Z_\mu$ be μ such functions having different poles, constants $C_1, C_2, \dots, C_\mu$ can be determined so that

$$C_1 Z_1 + C_2 Z_2 + \dots + C_\mu Z_\mu$$

is reproduced unchanged when continued over any closed path in F . As a closed path in F corresponds to a path joining two congruent points in the ζ -plane, this means that $\sum C_i Z_i$ is automorphic for the group Γ .

Among the properties of $\phi(\zeta)$ may be mentioned the following.

It takes any given value at μ points of the fundamental

region P , μ being the number of poles of $\phi(\zeta)$. From this follows that the region P can be conformally represented on an ordinary μ -leaved Riemann surface F of genus p by means of the function $z = \phi(\zeta)$. Any automorphic function of ζ is accordingly an algebraic function on the Riemann surface F . On the other hand every algebraic function of z , $W[z(\zeta)]$, is an automorphic function of ζ , so that the totality of automorphic functions belonging to the group Γ coincides with the totality of algebraic functions on the surface F . It follows from this that between any two functions ϕ_1, ϕ_2 which are automorphic for the group Γ an algebraic relation exists, $G(\phi_1, \phi_2) = 0$. If this relation is irreducible, then every automorphic function belonging to the same group is rationally expressible in terms of ϕ_1 and ϕ_2 . In particular, if $p = 0$, every such function is rationally expressible in terms of any function having a single pole in the fundamental region. Such a function is called a *principal function*. In general, a principal function is one which takes a given value the least possible number of times within the fundamental region.

It is evident from what precedes that every algebraic function on the Riemann surface F which is in general a many valued function of z , is single valued when expressed in terms of ζ . Every Riemann surface can be associated with a group Γ (in fact with more than one group), so that every algebraic function can be expressed as a uniform function. In the language of geometry, the coördinates of a point on any algebraic curve can be expressed as uniform (automorphic) functions of a variable parameter. To establish this result is one of the main problems of the book, or more explicitly stated: given any Riemann surface of genus p and n arbitrarily assigned points on it, $e_1, e_2, \dots, e_n$, to investigate the existence of a function $\zeta(z)$ which branches at the given points and which can mediate the representation of the Riemann surface on a polygon suitable for the fundamental region of a limit circle group and having vertices corresponding to e_i which are fixed points for substitutions of periods $l_1, l_2, \dots, l_n$, these being any integers > 1 , or infinite.

Since $\phi(\zeta)$ takes any given value μ times in every polygon which is congruent to the fundamental region P , and since these polygons crowd together in infinite number about the limit points of the group, it follows that $\phi(\zeta)$ takes any given value an infinite number of times in the immediate vicinity of

such a point which is accordingly a point of essential singularity for the function. If these limit points fill a curve, the limit curve, everywhere dense, this curve forms a natural boundary for the function over which it cannot be analytically continued.

Corresponding to the classification of the discontinuous groups, an analogous classification of the automorphic functions is given.

I. *Cyclic functions*, belonging to cyclic groups.

If the group is elliptic of period l , the principal function is the algebraic function $[(\zeta - \zeta_0)/(\zeta - \zeta'_0)]^l$. If the group is

parabolic, the principal function is $e^{\frac{2\pi i}{l} \frac{1}{\zeta - \zeta_0}}$. If hyperbolic, or loxodromic, the automorphic functions are the elliptic functions of $\log [(\zeta - \zeta_0)/(\zeta - \zeta'_0)]$.

II. *Elliptic functions*. The corresponding groups are the parabolic rotation groups.

III. *Fundamental circle functions*. These belong to groups whose limit points lie on a circle. If the circle is everywhere dense with fixed points, we have the special, but highly important, class of functions existing only within the circle and having the circumference for a natural boundary.

IV. *Automorphic functions in general, without fundamental circle*. This last and most extensive division of functions does not at present admit of precise classification owing to lack of knowledge of the non-rotation groups to which they belong. Three main subclasses are given, however, corresponding to the subdivisions of the groups mentioned above.

The inverse problem of determining the nature of ζ as a function of z in the Riemann surface F is next considered. When z describes a closed path in F , ζ is either unchanged or undergoes a linear transformation. It is infinitely many valued, and is accordingly called a polymorphic function. The important question arises as to whether in case any Riemann surface whatever is given, a polymorphic function exists, having the properties of the function ζ and capable of representing the Riemann surface on a polygon (or other region) suitable as the fundamental region for a group of linear substitutions. The full discussion of this problem is reserved for the remaining part of the volume which is not yet published, the author delaying on this point only to show that the polymorphic function ζ satisfies a differential equation of the third

order obtained by equating the "Schwarzian derivative" of ζ to a certain function which is algebraic on the surface F .

From a consideration of automorphic functions of ζ we pass on in the second chapter to the consideration of forms belonging to groups of genus zero. The variable ζ is separated into the quotient of two variables ζ_1, ζ_2 . Instead of the group Γ of substitutions of the form (6), the corresponding group of homogeneous substitutions

$$(7) \quad \zeta'_1 = \alpha_k \zeta_1 + \beta_k \zeta_2, \quad \zeta'_2 = \gamma_k \zeta_1 + \delta_k \zeta_2$$

is introduced. The course of the investigation is directed towards the construction of binary forms in ζ_1, ζ_2 , having the automorphic character. This treatment of the problem from the point of view of binary forms is characteristic of the methods of Klein as previously employed in the Icosahedron theory and the Modular Functions.

An automorphic form is defined as a homogeneous, non-branching function of ζ_1, ζ_2 of dimension d (d being any rational number) which is reproduced multiplied by a constant μ when the variables describe any continuous series of admissible value such that the end values ζ'_1, ζ'_2 are congruent to the initial values with respect to the homogeneous group of uni-modular substitutions (7).

That is,

$$\phi(\alpha_k \zeta_1 + \beta_k \zeta_2, \gamma_k \zeta_1 + \delta_k \zeta_2) = \mu_k \phi(\zeta_1, \zeta_2).$$

The quotient of two forms of like dimensions and the same multipliers is evidently an automorphic function of $\zeta_1/\zeta_2 = \zeta$.

Among the various forms that could be constructed for a given group ($p = 0$) two are of especial interest, the *principal form*, and the *prime form*. The former is defined by the equation

$$\phi_{-2}(\zeta_1, \zeta_2) = -\frac{1}{\zeta_2^2} \frac{dz}{d\zeta},$$

in which z is a principal function, that is, an automorphic function of ζ which takes a given value but once in the fundamental region. The principal form is absolutely invariant, is of dimension -2 , has one pole of order 2 at the point where $z = \infty$, and vanishes of order $1 - 1/l_k$ at a vertex ϵ_k of the

fundamental region which is a fixed point for a substitution of period l_k . It has no other poles or zeros.

If e_k is the value of z at $\zeta = \epsilon_k$, then $(z - e_k)$ vanishes of the first order at $\zeta = \epsilon_k$ and at all equivalent points: Hence the product

$$(8) \quad \phi_{-2} \prod_{k=1}^n (z - e_k)^{-\left(1 - \frac{1}{l_k}\right)},$$

in which ϕ_{-2} is the principal form, does not vanish at the vertices, and is zero of order

$$\sum_k \left(1 - \frac{1}{l_k}\right) - 2 = -\frac{2}{\nu}$$

at the pole ζ_0 of ϕ_{-2} . Accordingly, if the expression (8) is raised to the power $\left(-\frac{\nu}{2}\right)$, the result vanishes of the first order at ζ_0 . Such a form, denoted by $z_2(\zeta_1, \zeta_2)$, is of dimension ν in ζ_1, ζ_2 , and is finite and different from zero at every point of the fundamental region except ζ_0 . It is called a prime form.

The form

$$z_1(\zeta_1, \zeta_2) = z(\zeta) \cdot z_2(\zeta_1, \zeta_2)$$

is likewise a prime form whose zero point in the fundamental region coincides with that of z .

The formula $az_1 + bz_2$ defines a binary family of prime forms, all of which behave like z_2 with respect to the substitutions of the homogeneous group, and whose zero point varies with the parameter a/b .

If (z, e_k) denote the particular form of this family which vanishes at the fixed corner ϵ_k of the fundamental region, then the form

$$Z_k(\zeta_1, \zeta_2) = V^{l_k'}(z, \epsilon_k)$$

is a non-branching form for the group. It is called the *ground form* belonging to the corner ϵ_k . In case of an elliptic point l_k is the period of the substitution, and for a parabolic point l_k may be any positive integer.

The effect of the substitutions of the homogeneous group upon any form ϕ_a is next determined. This is accomplished by calculating the multipliers μ_k which arise from the generat-

ing substitutions. From this set of multipliers, called a multiplier system, the multiplier for any other operation of the group is obtained by a very simple formula.

The number of theoretically possible multiplier systems having been determined, it is then proved that automorphic forms of any allowable dimension d exist for every such system. This is done by first showing that every automorphic form can be expressed as a product of prime forms and ground forms, and that this product can be so chosen that the system of multipliers will coincide with any given possible system. In particular, the forms with the multiplier system $\mu_k = 1$ exist for every group Γ and for every integer dimension d . They are absolutely unchanged by substitutions of the group, and for this reason are called *proper* automorphic forms.

The variables ζ_1, ζ_2 when regarded as depending on z_1, z_2 are polymorphic forms in the latter. On multiplying ζ_1 and ζ_2 by certain forms in z_1, z_2 polymorphic forms are obtained which are of zero dimension and behave like ζ_1, ζ_2 with respect to substitutions of the group. Analytic expressions for these various forms are obtained, and differential equations of the second order which they satisfy are deduced.

Some of the foregoing results are illustrated by means of the hypergeometric function.

The third chapter treats of the Poincaré series with a detailed consideration of the case $p = 0$. This subject, which is one of the most important and fundamental in the entire theory of automorphic functions, is handled in a very felicitous and attractive manner.

"Poincaré series" is the name given in the present work to the series

$$(9) \quad \phi(\zeta_1, \zeta_2) = \sum_k \mu_k^{-1} H(a_k \zeta_1 + \beta_k \zeta_2, \gamma_k \zeta_1 + \delta_k \zeta_2),$$

in which $H(\zeta_1, \zeta_2)$ is any rational homogeneous function of ζ_1, ζ_2 , and μ_k is the multiplier, of any allowable system, corresponding to the transformation (7). The function ϕ thus defined is an automorphic form for the given group with the multiplier system μ_k .

If the above series be divided by ζ_2^d , d being the dimension of ϕ , and the multiplier system $\mu = 1$ taken, the resulting formula

$$\Theta(\zeta) = \sum_k H\left(\frac{a_k \zeta + \beta_k}{\gamma_k \zeta + \delta_k}\right) (\gamma_k \zeta + \delta_k)^d,$$

with d a negative even integer, is the "theta" series introduced by Poincaré, and is the principal element out of which he constructs the analytic theory of the automorphic functions. This function is reproduced, multiplied by a factor, by any substitution of the group; that is,

$$\Theta\left(\frac{\alpha_i\zeta + \beta_i}{\gamma_i\zeta + \delta_i}\right) = (\gamma_i\zeta + \delta_i)^{-d} \Theta(\zeta),$$

while the corresponding form ϕ is entirely unchanged.

The series (9) is proved by two methods, both due to Poincaré, to be convergent. The first proof establishes the convergence for a group with one limit curve, or an infinite number of them, and for all dimensions $d \leq -4$. This is the most general case and includes all others.

Under what conditions the Poincaré series is convergent for values of d greater than -4 is a problem not yet completely solved, but the case $d = -2$ receives somewhat extensive consideration. It is shown that for this value of d the series is no longer absolutely convergent whenever the group has one or more limit curves; but that for fundamental circle groups which are not limit circle groups, and for certain other groups without limit curves, the series is unconditionally and uniformly convergent. These results, which are largely due to the investigations of Burnside and Schottky, lead the authors to suggest with Burnside that the theorem is possibly true for all groups without limit curves. It is worthy of remark, also, that groups exist (for every genus p) for which the series of dimension $d = -1$ is absolutely convergent.

The second proof is given for the case of a fundamental circle group, and establishes the convergence of the series (9) for all dimensions d less than -2 .

The quotient of two Poincaré series of like dimension d and with the same multiplier system is an automorphic function of $\zeta = \zeta_1/\zeta_2$. Thus the Poincaré series affords a simple and elegant formula for the construction of functions belonging to the group.

One serious obstacle in the way of using this series arises from the possibility of its identical vanishing. That this possibility is not an imaginary one is shown by proving that identically vanishing series exist, and in infinite number for every dimension d . The difficulty is avoided by the construction of

series having one or more poles in the fundamental region. This is effected by assigning poles to the function H which is used in generating the series. The case of a single pole in the fundamental region being the simplest one is considered in detail. It is proved that such a series can always be constructed, and the details of the method, and the actual form of the series are given except in the case of absolute automorphic forms of dimension -2 which must have at least two poles. It is shown that the pole can be taken at any point in the polygon net, the parabolic fixed points alone excepted.

The one-pole series is now placed in the foreground. On the introduction of a suitable multiplying constant, it becomes discontinuous at the pole $\zeta_1 = \xi_1, \zeta_2 = \xi_2$ like $1/(\zeta_1 \xi_2 - \zeta_2 \xi_1)$. The series so normalized is called an *elementary form* and is denoted by $\Omega(\zeta_1, \zeta_2; \xi_1, \xi_2)$. It is worthy of notice that this elementary form, when regarded as depending on ξ_1, ξ_2 , is of dimension $-d-2$, and in its manner of becoming discontinuous behaves like an automorphic form with inverse multiplier system μ_k^{-1} . In order that Ω may be an automorphic form in ξ_1, ξ_2 it is necessary that, when expressed as a product of prime forms and ground forms, it be of dimension -1 in the former. It also has the same property when regarded as a form in ζ_1, ζ_2 , which fact leads to the expression of Ω in this case by a very elegant formula [(12), page 201].

Except for the special case just mentioned, Ω is not automorphic in ξ_1, ξ_2 . It has, however, certain properties as a function of these variables, which lead to the important conclusion that every automorphic form which vanishes in the parabolic points can be expressed in the form of a Poincaré series provided the dimension d is one for which that series is convergent. That automorphic forms exist which are not expressible as Poincaré series is shown by the occurrence of such a case for $d = -1$ in the modular group.

This result is further generalized by showing that every automorphic form with arbitrary poles none of which occur at a parabolic point, is expressible as a Poincaré series plus an integral automorphic form.

The fourth and last chapter generalizes the results to the case $p > 0$. As in this case the automorphic functions are no longer expressible in terms of a principal function z , the simple prime form for $p = 0$ must be replaced by the transcendental prime form which enters into the theory of abelian functions.

The Ritter prime form introduced here, while not so simple an analytic expression as the Klein prime form, has the advantage over the latter of not vanishing at the branch points of the Riemann surface F .

Klein's prime form $\Omega(x, y)$ is defined as the limit of the expression

$$\sqrt{-(x, dx)(y, dy)} e^{-\Pi_{x, y}^{x+dx, y+dy}}$$

for $dx = 0, dy = 0$. Here $\Pi_{x, y}^{x+dx, y+dy}$ is the normal integral of the third kind on the m -leaved Riemann surface, while (x, dx) is written for brevity to represent the homogeneous differential expression $x_1 dx_2 - x_2 dx_1$.

Ritter's prime form $P(z, e)$ is

$$\Omega(z, e) \sqrt[m]{\frac{z_2}{\Omega(z, \infty_1) \Omega(z, \infty_2) \cdots \Omega(z, \infty_m)}}$$

in which e is any point of the Riemann surface and $\Omega(z, \infty_i)$ is Klein's prime form with the path of integration extending to the point at infinity in the i th sheet of the surface, while z_2 is one of the homogeneous variables into which $z = z_1/z_2$ is separated. The prime form thus defined is everywhere continuous and has only a single zero of the first order at $z = e$. It is, however, in the periodic property of this prime form with respect to substitutions of the group Γ that its superiority over the Klein form is especially marked, for while the latter is reproduced multiplied by a function of z , the former has a multiplier independent of z .

Expressions are next obtained in terms of the Ritter prime form for the polymorphic forms ξ_1, ξ_2 whose quotient gives the variable ξ in terms of which z is an automorphic function.

Just as in the case $p = 0$, the variable ξ as a function of z satisfies a differential equation of the third order obtained by equating the Schwarzian derivative to a certain function R which is rational and algebraic on the Riemann surface, and which depends on $n + 3p - 3$ arbitrary constants, the accessory parameters of the differential equation. The form of the function R is determined for three cases of special interest, viz:

1. When the group Γ is of character $(1, n)$.
2. When it is hyperelliptic of character $(p, 0)$.
3. For the general case of character $(3, 0)$.

The polymorphic forms ζ_1, ζ_2 satisfy the differential equation of the second order

$$\frac{d^2 F}{dx^2} + RF = 0,$$

in which R is the function occurring in the differential equation of the third order for ζ .

The general multiplicative forms, that is, the automorphic forms which are reproduced with multiplying constants by the operations of the group Γ , are now taken under consideration, the main result arrived at being the theorem that such forms always exist for every integer dimension d and for every theoretically possible multiplier system. Every form ϕ of this kind is expressible as a product of ground and prime forms giving the zeros of ϕ , divided by a product of prime forms giving the poles of ϕ . As in the case $p = 0$, the ground form is a properly chosen root of a prime form which vanishes at one of the fixed corners of the fundamental polygon. By prime form is here meant the product of the Ritter form by an exponential factor whose exponent is an abelian integral of the first kind on the Riemann surface, such an exponential factor being the most general expression for a multiplicative form without poles or zeros.

Among the automorphic forms belonging to a group of genus p those of dimension -2 and multiplier system 1 which are free from poles have particular interest. Denoting such a form by $\Phi_{-2}(\zeta_1, \zeta_2)$ it is observed that the integral $\int \Phi_{-2}(\zeta_1, \zeta_2)(\zeta, d\zeta)$ is of dimension zero and hence a function of ζ (and therefore of z) which is everywhere finite on the Riemann surface F . It is accordingly an abelian integral of the first kind. As there are p linearly independent integrals of the first kind it follows that the same number of linearly independent functions Φ_{-2} exist. The number of linearly independent forms ϕ of given dimension and multiplier system is $t - p + \sigma + 1$, in which t is the number of zeros of ϕ , and σ is the number of linearly independent forms Φ_{-2} which vanish in these t zero points. This is clearly only another form of the Riemann-Roch theorem.

The expression of automorphic forms by means of the Poincaré series next demands attention. It is to be remarked first of all that in order to insure the convergence of this series, the multipliers μ_k were assumed unimodular. If a form ϕ' has

multipliers which are not unimodular, it can be expressed in the form

$$\phi' = e^{\sum c_i w_i} \phi,$$

in which ϕ is a form with unimodular multipliers and the w_i are normal integrals of the first kind on the Riemann surface F . A generator of Γ which corresponds to a crossing of a canonical period path in F will then reproduce ϕ' with a multiplier $e^{c_i \mu_i}$, p_i and μ_i being the corresponding moduli of periodicity of w_i and the multiplier of ϕ respectively. By a proper choice of the c_i it is evident that any desired multiplier system may be assigned to ϕ' .

In order that the form ϕ be expressible as a Poincaré series it is further necessary that it vanish in the parabolic fixed points. After a somewhat lengthy, but interesting analysis, a conclusion is reached similar to that in case $p = 0$, namely — every unimultiplicative form of dimension d satisfying the convergence condition, which vanishes in the parabolic points, and only such a form, is expressible as a Poincaré series; and every automorphic form with arbitrary poles, none of which occur at parabolic points, is expressible as a Poincaré series plus an integral automorphic form.

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LORIA'S SPECIAL PLANE CURVES.

Spezielle algebraische und transcendente ebene Curven, Theorie und Geschichte. Von Dr. GINO LORIA. Autorisierte, nach dem italienischen Manuscript bearbeitete deutsche Ausgabe von FRITZ SCHÜTTE. Leipzig, B. G. Teubner, 1902. 8vo., xxi + 744 pp. + 17 plates containing 174 figures.

ABOUT ten years ago the Royal Academy of Sciences at Madrid offered its triennial prize to be awarded upon the thirty-first of December, 1894, for "An ordered account of all curves of any kind which had received special names, and a further short account of their form, their equations, and their inventors." To this prodigious question no response seems to have come. Three years later the question was repeated. Professor Loria presented his researches, which were received

in a most flattering manner, as well they might be. These researches, in a somewhat altered form, carefully worked out and translated into German, form the contents of the large volume here under review.

Although the work is encyclopædic and all-inclusive so far as it goes, it does not cover the entire question set by the Academy at Madrid. Those curves, which are made up of arcs of other curves, and which therefore, though often of great use in applied mathematics or in the theory of the fine arts, are not representable by a single analytic equation, have purposely been omitted. No reference has been made to curves in space. The investigation of these must be left for a separate volume, which it is to be hoped Professor Loria will have time and inclination to write. In all other respects the author has covered the immense field of special curves whether named or not named. Thus, while in one way he has failed to compass the extent indicated in the announcement of the prize question, in another way he has exceeded the limits.

As the investigations on special curves often long antedate the *Jahrbuch* and lie hidden in doctors' theses, in essays, addresses, letters of a private correspondence, and in larger treatises where they are introduced only for some immediate special purpose, there can be little doubt — as the author himself modestly states — that some of the curves have not been attributed to their earliest inventors and that others may have been overlooked entirely. To find most of the special curves is not easy; to find them all is impossible. But probably no one person will detect more than a few corrections or additions to this work of Professor Loria, who has made such a profound and long-continued study of the field. Essential improvements must be the result of years and of many collaborators.

The method of treatment pursued by the author is, with the rarest exceptions, analytic. In the case of some curves, like the cissoid and conchoid, which date back to antiquity, a limited amount of geometric reasoning has been employed. For the most part, however, even these oldest curves have been handled analytically by the methods of rectangular or polar coördinates, or both. This method of treatment is to be commended, for only in this manner could be given anything approaching such a complete and systematic treatment as would interest modern readers. The large number of foot-notes, referring to the places where the original treatment of the curves may be

found, will be of invaluable service to those who delight in antiquarian methods or research. Most readers will find in the volume before us enough to satisfy them, except where the author has preferred to refer to some modern work in which a special curve is fully treated, rather than to give that treatment in detail. For example, the cardioid possesses numerous properties, of which only a few are given in the text, leaving the others to be studied in the elaborate thesis of Dr. R. C. Archibald.

Professor Loria's book will appeal in different degrees to a great variety of readers. That everything in the work should be interesting to a large number is inconceivable. Despite the careful and logical arrangement one can not fail to perceive that the problems are distinct and subject to distinct methods of treatment. The theory of special curves must necessarily be a collection of special theories, each applicable in general only to its particular curve. To this statement there are a few exceptions. If the curve happens to be unicursal, that is, expressible rationally in terms of a parameter, there are general theorems and methods which apply. Usually, however, one finds that he has at his disposal no other mathematical implements than elementary analytic geometry and the first principles of the differential and integral calculus. There must be some special motive or no one would read the book through as he might read a work on a more connected field of mathematics. On the other hand there is much here which will interest every one, especially one who teaches the elements of analytic geometry or of calculus. Every teacher knows the value of choosing problems which have interested mathematicians from early times down to the present day. Such problems are often easier for students to handle than the artificial ones invented purposely for exercises. The early mathematicians were not refined experts in analysis, having no great theories on which to draw. So they were forced to invent simple curves with simple properties immediately connected with simple geometric constructions, and these they treated by elementary means. The result is that this work done early in the development of mathematical science furnishes problems valuable in mathematical teaching.

This material has been collected by Professor Loria and put into a form easily accessible to all. For example the chapter on the cissoid of Diocles contains, in addition to the deduction of the equation of the curve, the following properties which,

taken collectively, form a series of interesting problems easy to prove by elementary methods. "The pedal of the parabola with respect to the vertex is a cissoid. The locus of points symmetric to the vertex of a parabola with respect to a (moving) tangent is a cissoid. If one parabola rolls externally upon an equal parabola the locus of the vertex is a cissoid. The polar reciprocal of a cissoid with respect to a circle about the cusp as center is a semi-cubical parabola. The envelope of the common chords of a cissoid and its (moving) circle of curvature is a similar cissoid of which the ordinates for corresponding abscissas are $\frac{1}{8} \cdot (\frac{7}{8})^2$ as great. The area between the cissoid and its asymptote is three times the area of the generating circle (Fermat; geometric interpretation by Huygens communicated to Wallis in a letter). The center of gravity of this area divides the distance between the cusp and the asymptote in the ratio 5 : 1. The volume generated by the revolution of the cissoid about its asymptote is equal to that of the ring generated by the same revolution of the generating circle (Sluse). The difference between the lengths of the cissoid and its asymptote when the ordinates y and $-y$ become infinite, approaches a finite quantity which may be evaluated by means of logarithms."

Here are eight theorems — four in analytic geometry, four in elementary calculus. These theorems are connected with such names as Diocles, Fermat, Huygens, Wallis, and Sluse; and the footnotes refer to places where these and further theorems may be found. Similar treatment is given to other curves including the cardioid, hypocycloid of three cusps, conchoid of Nicomedes, catenary, and tractrix. The student might well make the acquaintance of some of these famous curves early in his career. They are certainly more interesting than purely artificial loci. Then too there are curves in Professor Loria's book which are discussed by means of hyperbolic and elliptic functions and offer good exercise in their manipulation.

Most of the important theorems are printed in heavy faced type. They stand out on the text with an emphasis more than italic and may easily be found by rapidly turning the pages. A valuable and pleasing feature of the work is the elaborate plots of the important curves. There are one hundred and fifty numbered figures; but, as many of the numbers are subdivided, there are actually about one hundred and seventy-five different plots. The volume ends with an index of names, an index of subjects, and a special index for the plates. This careful atten-

tion to the form in which the volume is offered to the public is especially welcome in this case as it greatly facilitates rapid reference to any particular curve.

We now pass to a more detailed examination of the work. The first section of three chapters and thirteen pages, deals with the straight line, circle, and conic sections. Naturally the treatment is cursory, being in fact merely a few historical notes upon the most important ideas such as the geometry of a ruler, of a parallel ruler, or of the compasses. The author then passes immediately to higher plane curves.

The second section contains fourteen chapters reaching to the ninety-fourth page of the text and treating of cubics in point coördinates. The classification into five types (Chapter I) and the general treatment of unicursal (rational) cubics (Chapter II) are the most straightforward, neat and elementary that we know. The author, relying on the apology that his treatment is encyclopedic, gives F. W. Newman's classification of cubics into forty-two subtypes, each bearing a botanical or architectural name. It seems as if in this case a reference to the original memoir would be sufficient. In its place might have been given an account of that omitted but interesting work of Möbius which treats curves by means of their projection on a sphere. The circular curves, the cissoid with its generalization, the cubical parabola and folium of Descartes, the oblique and right strophoid, Sluse's conchoid, Rolle's curve, and the various witches, fill up the main part of the section. The two concluding, like the two beginning chapters, are the most interesting. They treat the cubic curves which serve for the trisection of an angle and the duplication of a cube, thus solving in a manner those two celebrated problems of antiquity, of which at least one is supposed to be of Pythian origin.

At this point we enter the only two serious complaints to be made against Professor Loria's work. First, there is practically no attention paid to line-coördinates, homogeneous or non-homogeneous. To be sure in earlier times these coördinates were unknown. It is only within the last century or half-century that they have become of importance. Hence there must be very few special curves defined by means of them. Yet in any extended treatise, even on special curves, these coördinates merit more attention than the author gives them — as they also do in elementary instruction where, they can serve as the most simple and important examples of envelopes. We

see in Professor Loria's work numerous cases of isolated or acnodal points; but isolated tangents seem to be lacking. This geometric phenomenon is however more worthy of note as it seems more peculiar and more confusing than in the case of points. This first objection may be only half-hearted, but not so our second.

A more important objection lies in the fact that no mention is made in the text or indexes of Abel or his celebrated theorem which yields beautifully and powerfully some of the most interesting properties of higher plane curves. That the theorem applies in general is no hindrance to its use on special curves. The lack of it is not particularly felt when dealing with rational curves of the third and fourth orders. The author obtains the congruences, which would be given by Abel's theorem, by another means — that of the rational representation. But in any work on algebraic curves, special or otherwise, the total omission of this theorem, which is perhaps the greatest single theorem in the subject, seems scarcely conceivable.

The third section extends to page 218 and contains sixteen chapters. To the first three chapters, on classification of quartics, unicursal quartics, and elliptic and bicircular quartics, the remarks made upon the first two chapters of the preceding section apply. The absence of Abel's theorem begins to be felt. The succeeding chapters treat of Perseus's spiral, Nicomedes's conchoid, three-cusped curves of the fourth order and in particular the hypocycloid, cartesian ovals, polyzomals of the fourth order, rational curves with a node at which the tangents to the two branches are coincident, conchals, cassinian ovals, curves with three double points at each of which both branches possess inflections, the mussel-line, a quartic for trisecting angles, quartics which may be obtained in simple manners from conics. The mussel-line, although not especially interesting for its geometric properties, deserves a passing note because among a number of other things it was invented by Dürer (1525) for use in the fine arts and was said by him "to serve many a useful purpose."

The titles of the chapters of the first three sections have been quoted so fully merely to indicate the vast amount of detail, the great care and exhaustiveness with which the volume before us has been compiled. There appear a number of names doubtless quite unknown to most readers. In many cases the curves themselves possess no very great interest and

are passed by with but short notice. Some of the chapters are less than two pages in length; others extend over more than a dozen. In the following sections only the more important or better known curves will be mentioned.

Section IV deals with special curves of order greater than four. The orders reach nine and even twenty-five in the case of some curves connected with lemniscate functions. Probably the most interesting curve is Watt's, derived from Watt's parallelogram. The other names such as astroids, scarabeans, nephroids, atriphtaloids probably represent so little to the mind as to be not worth quoting.

To this point the curves have so been defined as to possess a definite order. The author next passes to curves which may have any desired order. Among these are the parabolas and hyperbolas of order n , the pearl curves, Cayley's general polynomials defined by $\sum_i \sqrt{U_i} = 0$, ovals, curves for dividing an angle in a given ratio, self-polar curves, and curves obtained by inversion (anallagmatic). There are at the end of the section chapters, which possess a greater interest, upon algebraic curves whose rectification depends on assigned functions. The author gives applications to curves of which the length may be expressed in terms of arcs of an ellipse or lemniscate. Taken in its entirety this fifth section is one of the longest in the book; it contains seventeen chapters and over one hundred and fifty pages.

The sixth section is the longest, with its twenty-five chapters and almost two hundred pages, and the subject, special transcendental curves, is one of the most interesting. Among others are found curves for squaring the circle, spirals of various species, the whole trochoid family whether the base upon which the circle rolls be straight line or circle, the tractrix, catenary, cross-ratio or W -curves of Klein and Lie, lines of Mercator and Sumner, curves assumed by one-dimensional elastic bodies, the herpolhode, and some other curves of mathematical physics. Here, too, is a short account of the crinkly or otherwise amorphous curves. Strangely enough there seems to be no treatment of the Γ -curve in the text and the name does not occur in the index. The omission appears extraordinary as that curve not only possesses interesting properties of itself, but aids in the representation of many functions among which may be cited Euler's integrals of the second kind. The whole section is a veritable treasury of problems of a purely geometric nature

solvable by the use of the simpler transcendental functions, and as such it ought to appeal to those teachers of calculus who prefer not to draw very largely on physics for their problems. There are also excellent examples in the use of intrinsic coördinates and an appendix is devoted to showing how to pass from an equation in intrinsic to the corresponding equation in cartesian coördinates. The intrinsic equations are certainly interesting and important, and it is to be hoped that they will find greater recognition now that Professor Cesàro's work has appeared in German under the title *Natürliche Geometrie*.*

The seventh and concluding section, in twelve chapters and about one hundred and twenty pages, treats the very general question of curves which may be obtained by certain laws from a given curve. The generalized tractrix, evolutes, involutes, glissettes, parallel and radial curves, caustics, pedals and anti-pedals, and finally the derivative and integral curves derived by $y = f'(x)$ and $y = \int f(x)dx$ where $f(x)$ itself is defined by an equation $F(x, y) = 0$, form some of the topics. The last is worthy of especial mention owing to the fact that in general the derivative and integral curves are discussed only in the special case where the function $f(x)$ is a polynomial in x .

Although we have reached the end of the book proper, a note of a dozen pages under the unpromising title: Review of the historical development of the theory of plane curves, contains for advanced students a subject of more interest and stimulus than the rest of the volume taken together. This subject is the author's recent work on a class of curves which he calls *panalgebraic* and which are in general transcendental. The last seven pages of the note form a summary of Professor Loria's memoir, originally published in the *Prager Berichte*, and more recently reprinted with additions in the second volume of *Le Matematiche Pure ed Applicate*. Owing to the restricted American circulation of these periodicals the researches on panalgebraic curves are probably not well known among us, and we may be permitted in closing this review to go into detail upon the note with which the author closes his book.

Very truly he says: "What we need is a theory which will include most if not all of the known transcendental curves." Such is his theory of panalgebraic curves: for there are only about a half-dozen known curves which fail to fall under this

* See Dr. Snyder's review in the BULLETIN for April, 1903, pp. 349-357.

classification. The author considers a function $F(x, y, y')$ algebraic in each of the three variables x, y, y' . Without loss of generality we may assume that the function F is rational and integral and that the polynomials in x and y , which serve as coefficients of the powers of y' , have no common factor $\phi(x, y)$. Write

$$F(x, y, y') \equiv \sum_{r=0}^{r=n} f_r(x, y) y'^{n-r} = 0. \quad (1)$$

If the order of the polynomial F regarded as a function of x and y be ν , then ν is the order of that polynomial $f_r(x, y)$ which has the highest order.

Definition: A curve which satisfies a differential equation of the form (1) is said to be *panalgebraic*. The degree of the curve is said to be n , the degree of the polynomial F in y' . The rank of the curve is said to be ν , the degree of the polynomial F in x and y .

It is evident that the panalgebraic curves are to be regarded as the members of families, each of ∞^1 members: for the integral of equation (1) contains an arbitrary constant. An algebraic curve may be regarded as a panalgebraic curve of degree zero. The following theorems are fundamental in the theory.

I. If a curve is panalgebraic when regarded as a locus, it is panalgebraic when regarded as an envelope, and conversely.

II. In a family of panalgebraic curves of degree n and rank ν there are n curves which pass through each point of the plane and ν which touch each line.

III. In a family of panalgebraic curves (n, ν) there are $m\nu + n\mu$ which touch a given algebraic curve of order m and class μ .

IV. The points of tangency of the tangents drawn from a given point P to a panalgebraic curve (n, ν) lie on an algebraic curve of order $\nu + n$, of which the given point P is an n -fold point.

V. The tangents drawn at the points of intersection of a panalgebraic curve (n, ν) and a given line p touch an algebraic curve of class $n + \nu$, of which the given line p is an ν -fold tangent.

VI. The cusps of a panalgebraic curve lie on an algebraic curve.

VII. The inflection tangents of a panalgebraic curve touch an algebraic curve.

VIII. The system of all panalgebraic curves is invariant under the group of all contact transformations.

IX. The orthogonal trajectories of a family of panalgebraic curves form a family of panalgebraic curves of the same degree and rank.

X. There exist for panalgebraic curves certain derived algebraic curves analogous to the steinerian and hessian. Hence:

XI. The inflection points of a panalgebraic curve lie on an algebraic curve.

XII. The cuspidal tangents of a panalgebraic curve touch an algebraic curve.

When one considers that some relatively simple panalgebraic curves, such as the epicycloid in case the radii of the two circles are incommensurable, and the herpolhode practically fill completely an entire region of the plane, it is difficult not to be surprised at the simplicity of the results. For example the logarithmic spiral is of degree 1 and rank 1. The application of theorem IV yields the following result: The points of tangency of the tangents drawn from a point P to a logarithmic spiral lie upon a conic which passes through P . The application of the same theorem to Lissajous's curves ($n = 2, \nu = 2$) yields: The points of tangency of the tangents drawn from a point P to a Lissajous's curve lie upon a quartic which passes twice through the point P . A similar result obtains for the epicycloid ($n = 2, \nu = 4$) except that the locus is a sextic. Two general facts, first the dual property (theorem I), second the invariance under contact transformations (theorem VIII), will surely add to the simplicity and interest of any future detailed theory of panalgebraic curves.

In conclusion we can not only express our thanks to Professor Loria for his volume on *special* curves and wish it success, but we can, with a pleasure the greater by contrast, recognize in him the founder of a theory of curves more *general* than any heretofore known. We can most heartily join in his closing hope that the present century will bring forth the much desired general theory and classification of transcendental plane curves. It is reasonable to assume that this work of his can not fail to be one of the most important advances toward the full realization of that hope.

EDWIN BIDWELL WILSON.

ÉCOLE NORMALE SUPÉRIEURE, PARIS,
December 30, 1902.

SHORTER NOTICES.

Théorie Élémentaire des Séries. Par MAURICE GODEFROY, Bibliothécaire de la Faculté des Sciences de Marseille, avec une préface de L. SAUVAGE. Paris, Gauthier-Villars, 1903.

La Fonction Gamma. Par M. GODEFROY. Paris, Gauthier-Villars, 1902.

THE text-book literature of European mathematics has, since Euler's day, been blessed with numerous and most excellent books treating the elementary convergent processes from the algebraic standpoint. In such books we find a systematic application of the algorithm of inequalities to such questions as the convergence of infinite series, products and continued fractions and, after Cauchy's day, uniform convergence and term-wise differentiation and integration of series.

These books which used to go by the generic title of books on algebraic analysis * afford an admirable introduction to a *theoretic* course in the calculus, in that they present its basal ideas in a form at once most concrete and comprehensible to the beginner.

It has been the custom of continental text-book writers on the calculus in recent years to incorporate such of this matter as relates to the more essential notions concerning infinite series in the initial chapters of the differential calculus. This course, while it has no doubt been dictated by necessity, seems from some points of view unfortunate.

It has thus come about that since Cauchy's *Analyse algébrique* few books of similar purpose have appeared in French and of these the only one that need here be mentioned, Tannery's *Théorie des fonctions d'une variable*, while admirable in every way and destined not soon to be supplanted, is perhaps too extensive for the beginner.

M. Godefroy has given us a book on the elementary theory of series that has much to recommend it. The exposition has the traditional French clearness and evinces a pedagogic insight that one would only expect in a writer accustomed to actual instruction.

* This name, apparently coined by Cauchy, is still used in Italy.

The book begins with a short (too short) account of irrational numbers defined by a "section" and establishes certain fundamental theorems due to Cauchy concerning the equality of the limits

$$\lim_{n \rightarrow \infty} (x_{n+1} - x_n) = \lim_{n \rightarrow \infty} (x_n/n), \quad \lim_{n \rightarrow \infty} (x_{n+1}/x_n) = \lim_{n \rightarrow \infty} \sqrt[n]{x_n}, \text{ etc.}$$

After a section on continuous functions of a single variable, series with positive terms are taken up and the powerful and elegant test of Kummer is established.

This remarkable criterion, which contains as special cases the geometric test, Raabe's test and the test of DeMorgan-Bertrand, admits of a very simple proof; * it necessitates, however, the construction of divergent scales of positive numbers. Our author, however, gives an independent deduction of the geometric test and after establishing the familiar theorems concerning conditionally convergent series and sundry theorems of Cauchy and Mertens concerning double series proceeds to the consideration of series whose terms involve a variable parameter.

The regional property of uniform convergence is defined and illustrated, unfortunately without any graphs, and Weierstrass's m -test is established. The rest of the book is devoted to power series and the more important functions defined by them.

The statement of Abel's theorem concerning the interval of convergence of a power series is open to the objection that our author fails to distinguish — although the distinction is made sharply enough in an earlier part of the book — between a suite of numbers that are always finite and one that has an upper and lower boundary.

The demonstration of the term by term differentiability of a power series, apparently due to Biehler † is very simple and has the merit that it goes back to first principles; this result might however have been more readily deduced from the continuity of such series.

Noteworthy features of the book are the proof of the existence theorem for homogeneous linear differential equations with analytic coefficients and the proof of the transcendence of the natural base e . The collection of excellent examples that fol-

* Cf. Cesàro: *Nouvelles Annales*, vol. 7, 3d series, p. 406, the proof in the text is however sufficiently simple.

† *Nouv. Annal.*, vol. 7, 3d series, p. 200.

lows each chapter will be found useful and the bibliography appended seems adequate.

The last chapter of the book is devoted to the gamma function and the auxiliary functions of Prym. It is condensed from *La fonction gamma: Theorie, Histoire, Bibliographie* of our author, and although the available analytic processes are necessarily restricted, the proofs are elegant and compact.

M. B. PORTER.

Höhere Analysis für Ingenieure. Von Dr. JOHN PERRY. Autorisierte deutsche Bearbeitung von Dr. ROBERT FRICKE und FRITZ SÜCHTING. Leipzig, B. G. Teubner, 1903. 8vo., viii + 423 pp.

It is interesting to note in what numbers the last few years have produced treatises on the differential and integral calculus which have for their aim the fulfilment of the needs of some special class of students. For instance, there are for students of physics the elaborate treatise of Boussinesq and the smaller book by H. A. Lorentz, for chemists the work of Nernst and Schoenflies recently translated into English by Young and Linebarger, for students of political economy a small primer by Irving Fisher, and for engineers the work of Perry. The question naturally arises whether so much subdivision in the study of calculus is necessary. It will prove a matter of serious inconvenience if each specialist must have a special course and a special book to suit his needs. There can be little doubt that the present tendency to this subdivision is due partly to the fact that mathematicians are apt to wish too selfishly to make their elementary courses strictly mathematical instead of practical, and in this respect we hope they will mend. A great part of the difficulty, however, is due to the inertness of the students of special branches, who wish to learn so much and only so much of calculus as appears to them necessary for their immediate needs. The short-sightedness of this attitude renders it dangerous. To-day many a chemist or economist, whose elementary education was finished a decade ago, complains of experiencing inconveniences because he did not study calculus. Perhaps before twenty-five years are past those who now are trying to learn as little of it as possible will be wishing that they had not tried to economize so much. In the earlier years of instruction time is not so precious as later, symbolic processes fix themselves more readily upon the mind which

later will be occupied more with thought. Thus a student who leaves his undergraduate course without a thorough knowledge of the purely formal manipulation of mathematics seldom can make good his loss.

It seems to us that one who studies Professor Perry's book will run just this risk of acquiring an insufficient formal knowledge of calculus. The author states that "the young engineer cannot be practiced too thoroughly in the mere simplification of algebraic and trigonometric relations, including those which contain complex quantities." The statement applies as well to the calculus and casts considerable doubt upon the efficacy of the author's arrangement of his book, in which he leaves such simple rules of operation as the differentiation of a sum, product, or quotient of two functions or of a function of a function, until the last chapter (p. 305) to be classed under the title, "More difficult exercises and theorems." To whom the work will be valuable is hard to say. So full is it of applications to mechanics, electricity and magnetism, thermodynamics, and elasticity as to be practically unintelligible to one who has not had considerable training as an engineer; and so little stress does it lay on the formal side of calculus that even the engineer, far from obtaining a working knowledge of the subject, would do well if he learned so much as "not to be afraid of calculus when he saw it."

We pass over the surprisingly few offenses against accuracy in demonstrations and definitions and turn our attention to a few features which might advantageously be copied in other elementary text-books on calculus. The first is the omission of indeterminate forms $0/0$, ∞/∞ , $0 \cdot \infty$, etc., of which the rigorous treatment is a matter of considerable difficulty. It is doubtful if students even understand what they are doing when they differentiate numerator and denominator until they find a determinate result. The whole subject may well be postponed until after treating the algebraic operations upon series. Secondly the introduction of the simpler differential equations and the methods of integrating them affords a pleasant and valuable change from the lengthy treatment of the formal integration of a rational function or a function rational in the domain $(x, \sqrt{ax^2 + bx + c})$ with which most text-books insist on tormenting the students notwithstanding that a table of integrals like Professor B. O. Peirce's serves every purpose. Meanwhile, despite the fact that almost every application of cal-

culus involves a differential equation, the students remain unable to treat even the equation of harmonic motion and completely ignorant of the necessity and significance of "determining the arbitrary constants." Thirdly the author introduces Fourier's series long before he reaches the chapter on "more difficult exercises and theorems" in which he first states the rule for differentiating a product. Curious as this arrangement may seem, we wish to use it to call attention to the value and the possibility of early introducing the expansion of a function into a Fourier's series. Mathematicians as a rule are so pre-occupied with contemplating the difficulties encountered in proving the expansibility of an arbitrary function into a Fourier's series that their students study calculus three hours a week for two full years, one hundred and eighty lessons, without hearing of any such series. Yet so valuable are these series in physics and astronomy that they might well be introduced, even crudely, as Professor Perry introduces them, early in the study of calculus. The actual determination of the coefficients in a case of the simpler functions is not difficult and furnishes a practical problem in integration.

Such are some of the advantages and disadvantages of Professor Perry's *Calculus for Engineers*. We can heartily recommend to any teacher the perusal of this book. In many places he will find his eyes opening to ways in which the treatment of difficulties, unsuspected and unappreciated by the student, may be replaced by questions of practical use and interest without endangering the career of any possible future pure mathematician. It will be only by a rational compromise on both sides that the inconvenience of the present tendency to different courses and different books for each class of specialists can be avoided without harm to any.

E. B. WILSON.

Leçons sur les Fonctions Méromorphes. Par ÉMILE BOREL. Recueillies et rédigées par LUDOVIC ZORETTI. Paris, Gauthier-Villars, 1903. 8vo., vi + 122 pp.

THE *Leçons sur les Fonctions Méromorphes* is a natural sequel to the author's *Leçons sur les Fonctions Entières* which appeared about three years ago. The results are in great part mere generalizations, to the case of functions possessing poles, of the results earlier obtained for functions possessing no singularities in the finite region of the complex plane. Considerable

attention is paid to numerical equalities and inequalities. The work draws its inspiration from Weierstrass, Du Bois-Reymond, and especially from Cauchy. It is distinctly of the modern French School, however, for with the exceptions of Mittag-Leffler and Lindelöf, who draw from the same source, the names that one meets are Picard, Poincaré, Hadamard, Borel and other less-known French mathematicians.

The chief topics treated are Mittag-Leffler's theorem that a meromorphic function is the sum of an integral (transcendental) function and a series of rational fractions; Weierstrass's theorem that a meromorphic function is the quotient of two integral functions; Hadamard's investigation on Taylor's series with applications to meromorphic functions and to the zeros of integral functions; the generalization of Picard's theorem to the statement that any meromorphic function must take on every value, except possibly two, an infinite number of times; and series of rational fractions, with ordinary and extraordinary distributions of poles, as applied to the study of meromorphic functions. Four notes, two of which Mr. Zoretti contributed, bring the subject so far up to date as to include some memoirs not published when the book went to press.

This little book, like the others, is of great value in putting us abreast of the present state of a particular branch of mathematics. So carefully has the matter been chosen and so clearly has Mr. Zoretti written that only a very limited knowledge of the theory of functions is presupposed. This pedagogical method is peculiarly French. It quickly places the student in absolute command of an advanced field so that he may commence his investigations at once without that vast auxiliary knowledge which some consider necessary. By such a work as these *Leçons* the whole mathematical world is given the same advantage.

E. B. WILSON.

NOTES.

THE April number (volume 25, number 2) of the *American Journal of Mathematics* contains the following papers: "The double-six configuration connected with the cubic surface, and a related group of Cremona transformations," by EDWARD KAS-

NER; "Untersuchungen über lineare Differentialgleichungen 4. Ordnung und die zugehörigen Gruppen," by SAUL EPSTEIN; "The logic of relations, logical substitution groups, and cardinal numbers," by A. N. WHITEHEAD; "On differential equations belonging to a ternary linearoid group," by F. E. ROSS; "On a certain group of isomorphisms," by J. W. YOUNG.

THE April number (second series, volume 4, number 3) of the *Annals of Mathematics* contains: "The cardioid and tricuspoid: quartics with three cusps," by R. C. ARCHIBALD; "Note on a partial differential equation of the first order," by E. D. ROE, JR.; "On a generalization of the set of associated minimum surfaces," by A. S. GALE; "Twisted quartic curves of the first species and certain covariant quartics," by H. S. WHITE; "On the characteristics of differential equations, Part I," by E. R. HEDRICK.

At a meeting held in Boston, April 18, an association of teachers of mathematics in New England was organized for the improvement of methods of mathematical instruction in schools and colleges. Professor T. S. FISKE made an opening address on methods of improving the teaching of mathematics and Mr. W. T. CAMPBELL, of the Boston Latin School, read a paper on "Observational geometry." The following officers were elected: President, Mr. E. H. NICHOLS; vice-president, Professor W. F. OSGOOD; secretary and treasurer, Mr. F. P. DODGE, Roxbury Latin School, Boston, Mass. The association has at present about one hundred and sixty members. Copies of the constitution and of the circular letter issued as a call for the recent meeting can be obtained from the secretary.

At the meeting of the London mathematical society on March 12, the following papers were read: "On the convergence of certain multiple series," by G. H. HARDY; "On certain sequences for determining the n th root of a rational number," by Mr. S. M. JACOB; "Note on the approximate calculation of the frequencies of a vibrating circular plate," by Professor H. LAMB; "On surfaces which have assigned families of curves as their lines of curvature," by Professor A. R. FORSYTH; "Note on a point in Hilbert's 'Grundlagen der Geometrie,'" by Mr. E. T. DIXON; "Extension of two theorems on covariants," by Mr. J. H. GRACE; "Note on double limits and on the inversion of a repeated infinite inte-

gral," by Professor T. J. P. A. BROMWICH; "On the representation of a group of finite order as an irreducible group of linear substitutions, and the direct establishment of the relations between the group characteristics," by Professor W. BURNSIDE.

THE section of the history of sciences of the international congress of historic sciences, which met in Rome in April, 1903, has decided to form an international commission on the organization of a congress of the history of sciences. The following officers have been chosen: President, J. TANNERY, Paris; secretaries, P. GIACOT, Turin, and G. LORIA, Genoa. Professor D. E. SMITH has been appointed the American representative. The next meeting of the congress will be held in Berlin, in September, 1906.

THE German association for the promotion of the teaching of mathematics and natural sciences held its twelfth general meeting at Breslau, June 1-5.

THE several American universities below offer during the academic year 1903-1904 advanced courses in mathematics as follows:

UNIVERSITY OF CALIFORNIA.—By Professor I. STRINGHAM: Logic of mathematics, two hours; Analytic non-euclidean geometry of space, two hours.—By Professor G. C. EDWARDS: Ordinary differential equations, three hours.—By Professor M. W. HASKELL: Theory of functions of a complex variable, three hours.—By Mr. A. W. WHITNEY: Theory of probabilities, two hours.—By Dr. D. N. LEHMER: Synthetic projective geometry, three hours.—By Dr. E. M. BLAKE: Line geometry, three hours.—By Dr. T. M. PUTNAM: Theory of numbers, three hours; Mathematical seminar, foundations of dynamics, two hours.

THE UNIVERSITY OF CHICAGO.—The following advanced mathematical courses, four hours weekly, are offered during the four quarters (*su*, *a*, *w*, *sp*) of the year beginning June 17, 1903:—By Professor E. H. MOORE: Theory of functions of real variables (*su*); Seminar (*a*, *w*, *sp*).—By Professor O. BOLZA: Theory of equations (*a*, *w*); Quaternions (*sp*); Elliptic functions, the Weierstrass theory (*a*); Applications of elliptic functions (*su*); Abelian functions (*w*, *sp*); Invariants (*su*).—

By Professor H. MASCHKE: Solid analytics (*sp*); Twisted curves and surfaces (*w, sp*); Invariants (*w*).—By Professor H. E. SLAUGHT: Advanced integral calculus (*aw*).—By Professor J. W. H. YOUNG: Solid analytics and determinants (*su*); conferences on the pedagogy of mathematics (*su*).—By Professor L. E. DICKSON: Theory of functions of a complex variable (*su*); Theory of functions (*a, w*).—By Mr. A. C. LUNN: The differential equations of mathematical physics (*su, sp*); Graphic methods in the teaching of elementary mathematics (*su*).—By Dr. S. EPSTEIN: History of elementary mathematics (*su, 2* hours).—By Professor K. LAVES: Analytic mechanics (*a, w*).—By Professor F. R. MOULTON: Introduction to celestial mechanics (*w, sp*); Selected chapters of celestial mechanics (*su*).

CORNELL UNIVERSITY.—By Professor L. A. WAIT: Advanced analytic geometry (plane and solid), three hours; Advanced differential calculus, three hours.—By Professor G. W. JONES: Higher algebra and trigonometry, three hours.—By Professor J. MCMAHON: Higher plane curves, two hours; Potential function, two hours; Quaternions, two hours; Theoretical mechanics, two hours; Mathematical theory of sound.—By Professor J. H. TANNER: Theory of equations, two hours; Invariants, two hours.—By Dr. J. I. HUTCHINSON: Theory of functions, two hours; Elliptic and abelian functions, two hours; Projective geometry, three hours.—By Dr. V. SNYDER: Advanced integral calculus, two hours; Line geometry, three hours.—By Dr. W. B. FITE: Differential equations, two hours; Theory of groups, three hours.

HARVARD UNIVERSITY.—By Professor J. M. PEIRCE: Quaternions; Theory and application of tetrahedral coördinates; † The algebra of logic; † Finite differences; † The calculus of probabilities; † Linear associative algebra.—By Professors W. E. BYERLY and B. O. PEIRCE: Trigonometric series, spherical harmonics, and the potential function. By Professor W. E. BYERLY: Calculus (second course); Introduction to modern geometry and modern algebra; † Dynamics of a rigid body.—By Professor W. F. OSGOOD: The theory of functions (first course).—By Professor M. BÔCHER: † Infinite series and products; Algebra; † Linear differential equations.—By Dr. C. L. BOUTON: † The theory of numbers; † The elementary theory of differential equations; † Geometric transformations.—By Mr. J. K. WHITTEMORE: Celestial mechanics; † Hydromechanics.

These courses will involve three lectures a week throughout the year, except those preceded by †, which involve about half this number of lectures. Professor Bôcher, Dr. Bouton, and Mr. Whittemore also offer courses in reading and research in Differential equations, Continuous groups, and Differential geometry respectively.

STANFORD UNIVERSITY.—Professor R. E. ALLARDICE: Invariants, two hours, first semester; Definite integrals, two hours, second semester; Geometry of three dimensions, two hours; Theory of functions, three hours.—By Professor R. L. GREEN: Advanced coördinate geometry, two hours; Theory of equations, three hours.—By Professor L. M. HOSKINS: Theory of attraction and the potential function, two hours.—By Professor G. A. MILLER: Elementary theory of groups, three hours; Theory of numbers, two hours, second semester; History of mathematics, two hours, first semester; Seminar in the theory of groups, two hours.—By Professor H. F. BLICHFELDT: Determinants, two hours, first semester; Non-euclidean geometry, two hours, second semester; Differential equations, three hours.

NORTHWESTERN UNIVERSITY:—*First semester*.—By Professor T. F. HOLGATE: Linear systems of conics, two hours.—By Professor H. S. WHITE: Theory of functions, two hours; Plane cubics and quartics, three hours. *Second semester*.—By Professor T. F. HOLGATE: Theory of numbers, two hours.—By Professor H. S. WHITE: Elliptic functions, three hours; Projective geometry of surface and twisted curves, two hours.

YALE UNIVERSITY.—By Professor JAMES PIERPONT: Advanced calculus, three hours; Projective geometry, three hours; Theory of functions of real variables, three hours.—By Professor P. F. SMITH: Continuous groups, three hours; Higher analysis for engineers, two hours, first semester.—By Professor H. A. BUMSTEAD: Problems in mathematical physics, two hours.—By Dr. A. S. GALE: Differential equations and function theory, three hours.—By Dr. H. E. HAWKES: Higher algebra, two hours; Elliptic functions, three hours.—By Dr. W. A. GRANVILLE: Differential geometry, two hours.—By Dr. E. R. HEDRICK: Calculus of variations, three hours.—By Dr. E. B. WILSON: Analytic mechanics, two hours; Introduction to mathematical physics, two hours; Non-euclidean geometry, two hours.

THE several foreign universities and technical schools below offer during the present summer semester courses in mathematics as follows :

UNIVERSITY OF BERLIN. — By Professor J. KNOBLAUCH : Analytic geometry, four hours ; Theory of partial differential equations, four hours ; Theory of ray systems, one hour. — By Professor R. LEHMANN-FILHÉS : Differential equations, four hours ; Exercises, one hour. — By Professor H. A. SCHWARZ : Integral calculus, four hours ; Exercises ; Applications of elliptic functions, two hours ; Calculus of variations, four hours ; Colloquium ; Seminar. — By Professor F. SCHUR : Introduction to the theory of ordinary differential equations, two hours. — By Dr. E. LANDAU : Theory of numbers, four hours ; Introduction to the theory of functions, four hours ; Theory of Riemann zeta-functions, one hour. — By Professor G. FROBENIUS : Theory of algebraic equations, four hours ; Seminar. — By Professor F. H. SCHOTTKY : Theory of abelian functions, four hours ; Seminar. — By Professor G. HETTNER : Theory of potential, two hours.

UNIVERSITY OF CZERNOWITZ. — By Professor O. TUMLIRZ : Mathematico-physical seminar, two hours ; proseminar, two hours.

UNIVERSITY OF GRAZ. — By Professor J. FRISCHAUF : Differential and integral calculus, and its application to geometry, five hours. — By Professor V. D. R. VON KOLLESBERG : Analytic and projective geometry of the plane, five hours ; Mathematical seminar, two hours. — By Dr. J. STREISSLER : Descriptive geometry, three hours.

UNIVERSITY OF INNSBRUCK. — By Professor O. STOLZ : Theory of functions of a complex variable according to Cauchy and Weierstrass, with exercises in the mathematical seminar, three hours ; Arithmetic, the theory of real numbers, with exercises in the mathematical seminar, four hours. — By Professor W. WIRTINGER : Higher algebra, three hours ; Abelian functions, two hours ; Seminar, two hours. — By Professor K. ZINDLER : Plane and solid analytic geometry, two hours ; Line geometry, three hours ; Seminar, one hour.

UNIVERSITY OF KÖNIGSBERG. — By Professor F. MEYER : Analytic geometry, four hours ; Exercises in analytic geometry,

one hour ; Introduction to higher geometry, four hours ; Exercises in higher geometry, one hour.—By Professor A. SCHOENFLIES : Theory of differential equations four hours ; Exercises in the mathematical seminar, two hours.—By Professor L. SAALSCHÜTZ : Determinants, two hours ; Gaussian and other interesting series, four hours.—By Dr. F. COHN : Applications of the potential theory, three hours.—By Dr. T. VAHLEN : Differential calculus, four hours ; Exercises in differential calculus, one hour.

UNIVERSITY OF MÜNSTER.—By Professor W. KILLING : Differential and integral calculus, I, three hours ; Synthetic geometry, four hours ; Selected topics from elementary mathematics, two hours ; Exercises in the proseminar, two hours ; Exercises in differential and integral calculus, one hour ; Supplement to analytic geometry, one hour.—By Professor R. VON LILIENTHAL : Analytic geometry, I, four hours ; Theory of determinants, with applications, four hours ; Exercises in the mathematical seminar, one hour.—By Dr. M. DEHN : Theory of numbers, four hours ; Higher mathematics for naturalists, three hours ; Exercises in the theory and application of elliptic functions, one hour.—By Dr. J. KARST : Introduction to descriptive geometry.

UNIVERSITY OF PRAG.—By Professor G. PICK : Differential equations, three hours ; Differential and integral calculus, two hours ; Seminar, two hours.—By Professor J. A. GMEINER : Analytic geometry, three hours ; Number congruences, two hours.—By Dr. W. WEISS : Elements of descriptive geometry, two hours.

UNIVERSITY OF VIENNA.—By Professor G. VON ESCHERICH : Elements of the differential and integral calculus, five hours ; Exercises, two hours ; Proseminar, one hour ; Seminar, two hours.—By Professor F. MERTENS : Theory of numbers (continuation), five hours ; Exercises in the mathematical seminar, two hour ; in the proseminar, one hour ; Theory of probabilities, three hours ; Mathematical statistics, three hours.—Professor G. KOHN : Synthetic geometry (continuation), four hours, with exercises, one hour ; Theory of invariants, with geometric applications (continuation), two hours.—By Dr. E. BLASCHKE : Introduction to mathematical statistics, II, three hours.—By Dr. DAUBLESKY VON STERNECK : Algebra, three

hours.—By Dr. K. CARDA : Selected topics from the theory of contact transformations, two hours.—By Dr. J. PLEMELJ : Theory of potential, with applications (continuation), two hours.

UNIVERSITY OF WÜRZBURG.—By Professor F. PRYM : Integral calculus, six hours ; Exercises on integral calculus, in the proseminar, two hours ; Selected chapters in the theory of functions, in the seminar, two hours.—By Dr. G. ROST : Descriptive geometry, II, four hours ; Analytic and synthetic geometry of conics, four hours ; Application of the infinitesimal analysis to geometry, four hours ; Selected chapters from elementary mathematics in the proseminar, two hours.

UNIVERSITY OF BASEL.—By Professor H. KINKELIN : Algebraic analysis, three hours ; Geometric applications of differential calculus, three hours ; Definite integrals, two hours ; Theory of probabilities, three hours.—By Dr. R. FLATT : Line geometry, two hours.

UNIVERSITY OF BERN.—By Professor J. H. GRAF : Bessel's functions, three hours ; Elliptic functions, three hours ; Function theory, two hours ; Differential equations, two hours ; Differential and integral calculus, two hours ; Mathematics of finance and insurance, two hours ; Seminar, with Professor Huber, two hours ; Seminar (insurance) with Professor Moser, one hour.—By Professor G. SIDLER : On elliptic arcs, whose difference is rectifiable, two hours.—By Professor G. HUBER : Introduction to the theory of algebraic surfaces, two hours.—By Professor E. OTT : Differential calculus, two hours ; Analytic geometry, I, two hours.—By Dr. A. BENTELI : Elements of descriptive geometry, four hours.—By Dr. L. CRELIER : Synthetic geometry of space, two hours.

UNIVERSITY OF GENEVA.—By Professor C. CAILLER : Differential and integral calculus (continuation), three hours ; Seminar, two hours.—By Professor H. FEHR : Descriptive and projective geometry, two hours ; Algebra ; General theory of equations, two hours ; Exercises in the calculus (with Professor Cailler), two hours, and in algebra and geometry, two hours.—By Dr. D. MIRIMANOFF : Dirichlet's problem, two hours.

UNIVERSITY OF LAUSANNE.—By Professor H. AMSTEIN: Differential and integral calculus, five hours; Exercises, two hours; Theory of elliptic functions, three hours; Calculus for naturalists, three hours; Selected topics from the integral calculus (definite integrals and series), two hours.—By Professor H. JOLY: Analytic geometry (continuation), two hours; Descriptive geometry (continuation), two hours; Outline of descriptive geometry, four hours; Theory of numbers, two hours.

UNIVERSITY OF ZURICH.—By Professor H. BURKHARDT: Linear differential equations, four hours; Vector analysis, two hours; Seminar, two hours; Mathematical treatment of periodic recurrences in nature, two hours.—By Professor A. WEILER: Analytic geometry, II, two hours; Descriptive geometry, three hours; Mathematical geography, two hours.—Map projections, two hours.—By Dr. F. KRAFT: General theory of assemblages, four hours.—By Dr. E. GUBLER: Theory of numbers, two hours; Political arithmetic, with exercises, two hours.

CARLSRUHE TECHNICAL HIGH SCHOOL.—By Professor R. HAUSSNER: Elementary and analytic geometry of the plane and space, three hours, with exercises, one hour; Synthetic geometry, two hours, with exercises, one hour.—By Professor A. KRAZER: Higher mathematics, I, six hours, with exercises, two hours.—By Professor F. SCHUR: Descriptive geometry, four hours, with exercises, four hours.—By Professor L. WEDEKIND: Higher mathematics, II, two hours.—By Dr. G. HAMEL: Elements of higher mathematics, four hours, with exercises, one hour.—By Dr. W. LUDWIG: Theory of projection, two hours, with exercises, two hours.

MUNICH TECHNICAL HIGH SCHOOL.—By Professor A. VON BRAUNMÜHL: Algebraic analysis, with exercises (continuation); projective geometry synthetically treated, with exercises (continuation); mathematico-historical seminar.—By Professor S. FINSTERWALDER: Higher mathematics, II, with exercises; non-euclidean geometry.—By Professor W. VON DYCK: Partial differential equations of mathematical physics; colloquium, with Professor Finsterwalder.—By Professor L. BURMESTER: Descriptive geometry, with exercises (continuation).—By Professor S. GÜNTHER: Theory of potential with its application to geophysics.

STUTTGART TECHNICAL HIGH SCHOOL. — By Professor W. BRETSCHNEIDER : Elementary mathematics. — By Professor K. MEHMKE : Descriptive geometry ; seminar — By Professor K. REUSCHLE : Differential and integral calculus ; Plane analytic geometry ; Seminar. — By Dr. E. WÖLFFING : Theory of curves ; Partial differential equations.

THE current numbers of the *Revue générale des sciences*, beginning with volume 14, number 2, contain a very interesting series of articles in which, under the caption "The evolution of mechanics," P. DUHEM discusses the relations of theoretical mechanics to physical theories. The titles of the successive papers will give some idea of the wide field covered : I. Les diverses sortes d'explications mécaniques ; II. La mécanique analytique ; III. Les théories mécaniques de la chaleur et de l'électricité ; IV. Le retour à l'atomisme et au cartésianisme ; V. Les fondements de la thermodynamique ; VI. La statique générale et la dynamique générale ; VII. Les branches aberrantes de la thermodynamique.

PROFESSOR P. GORDAN, of Erlangen, has been made an honorary member of the University of Juriev (Dorpat). Dr. I. IVANOF and Dr. W. STANIEVICH have been made professors of mathematics at the St. Petersburg Polytechnic School. Dr. L. NATANSON has been promoted to a full professorship of mathematics at the University of Cracow.

PREBENDARY W. A. WHITWORTH, formerly fellow of St. John's College, Cambridge, has been appointed Hulsean lecturer for 1903.

MR. HERBERT KNAPMAN, first Smith's prizeman this year, has been appointed assistant lecturer in mathematics at University College, Reading, England.

MR. W. H. WAGSTAFF delivered the Gresham lectures on geometry this year, April 28–May 1.

PROFESSOR G. DARBOUX has recently been elected member of the bureau of longitudes.

PROFESSOR W. F. OSGOOD, of Harvard University, has been promoted to a full professorship of mathematics.

DR. ARNOLD EMCH, of the University of Colorado, has been promoted to a full professorship of graphics and mathematics.

DR. C. A. NOBLE has been promoted to an assistant professorship of mathematics at the University of California.

At Lehigh University Professor ALEXANDER MACFARLANE delivered, April 20-30, a course of six lectures on the British mathematicians Kirkman, Babbage, Whewell, Dodgson, Stokes, and Rayleigh.

MR. F. G. WREN has been advanced from an assistant professorship to a professorship of mathematics at Tufts College.

PROFESSOR J. W. MASON, of the College of the City of New York, will retire from service at the end of the present year.

DR. N. M. FERRERS, for a long time one of the editors of the *Quarterly Journal of Mathematics*, died on January 31, 1903.

PROFESSOR JOSIAH WILLARD GIBBS, of Yale University, died at New Haven, April 28, 1903, of heart disease. He was born in New Haven, February 11, 1839, and graduated at Yale in 1858. After receiving the doctor's degree in 1863 from the same institution, he studied in Paris, Berlin, and Heidelberg. In 1871 he was appointed to the professorship of mathematical physics at Yale which he held to the time of his death. Professor Gibbs was a member of the Royal Society of London, of the National Academy of Sciences, and of many other learned bodies. Quite recently he joined the AMERICAN MATHEMATICAL SOCIETY. He was an authority of the first rank in thermodynamics and in the application of vector analysis to physical problems.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMODEO (F.). Appunti e riposte. Lettera aperta ad un geometra italiano. Napoli, 1902. 8vo. 10 pp.

—. Dai fratelli di Martino a Vito Caravelli. Napoli, 1902. 8vo. 64 pp.

BIBLIOGRAPHIE der deutschen naturwissenschaftlichen Litteratur. Herausgegeben im Auftrage des Reichsamtes des Innern vom deutschen Bureau der internationalen Bibliographie in Berlin. Vol. 3: 1903-4. Abteilung 1: Mathematik, Mechanik, Physik, Chemie, Astronomie, Meteorologie. No. 1. Jena, Fischer, 1903. 8vo. 48 pp. M. 9.00

- BRADSHAW (J. W.). The logarithm as a direct function. 4to. (*Annals of Mathematics* (2) 4, pp. 51-62.)
- BRAUNMÜHL (A. von). Vorlesungen über Geschichte der Trigonometrie. Teil 2: Von der Erfindung der Logarithmen bis auf die Gegenwart. Leipzig, Teubner, 1903. 8vo. 11 + 264 pp. M. 10.00
- BUCHERER (A. H.). Elemente der Vektoranalysis; mit Beispielen aus der theoretischen Physik. Leipzig, Teubner, 1903. 8vo. 6 + 91 pp. M. 2.40
- COUTURAT (L.). See LEIBNIZ.
- DINGELDEY (F.). See ENCYKLOPÄDIE.
- ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. III: Geometrie, herausgegeben von W. F. Meyer. Teil 2, Heft 1: F. Dingeldey, Kegelschnitte und Kegelschnittssysteme. Leipzig, Teubner, 1903. 8vo. Pp. 1-160.
- ESTANAVE (E.). Essai sur la sommation de quelques séries trigonométriques. Paris, Hermann, 1903. 8vo. 112 pp. Fr. 6.00
- GULDBERG (A.). See LIE (S.).
- INTERNATIONAL CATALOGUE of scientific literature. First annual issue [for 1901]. A. Mathematics. Published for the International Council by the Royal Society of London. London, Harrison. Vol. VII: 1902 (November). 8vo. 16 + 201 pp.
- JAHRHAUS (K.). Das Verhalten der Potenzreihen auf dem Konvergenzkreise, historisch-kritisch dargestellt. (Progr.) Ludwigshafen, 1902. 8vo. 56 pp.
- JOUFFRET (E.). Traité élémentaire de géométrie à quatre dimensions et introduction à la géométrie à n dimensions. Paris, Gauthier-Villars, 1903. 8vo. 30 + 216 pp. Fr. 7.50
- KASNER (E.). The cogredient and digredient theories of multiple binary forms. 4to. (*Transactions of the American Mathematical Society* 4, pp. 86-102.)
- . The generalized Beltrami problem concerning geodesic representation. 4to. (*Transactions of the American Mathematical Society* 4, pp. 149-152.)
- LEIBNIZ. Opuscules et fragments inédits. Extraits des manuscrits de la bibliothèque royale de Hanovre par L. Couturat. Paris, Alcan, 1903. 8vo. 689 pp. (Collection historique des grands philosophes.)
- LIE (S.). Ueber Integralinvarianten und Differentialgleichungen. (Ausgegeben von A. Guldberg und K. Störmer im Auftrage der k. Gesellschaft der Wissenschaften zu Kristiania und auf Kosten des Fridjof Nansen Fond.) Kristiania, 1902. 8vo. 73 pp. K. 2.00
- LINDOW (M.). Die Nullstellen des allgemeinen Integrals der Differentialgleichung für die zugeordneten Kugelfunktionen. (Diss.) Halle, 1902. 8vo. 65 pp.
- LINNEBORN (J.). Die Fokaleigenschaften der Gebilde zweiter Ordnung in der Riemannschen Raumform. (Diss.) Münster, 1902. 8vo. 26 pp.
- LIPPITSCH (K.). Die Unverträglichkeits-Relation des Satzes vom goldenen Schnitte mit dem Gesetze der rationalen Indices, nachgewiesen am Rautendreissigflächner und regelmässigen Pentagondodekaeder. (Progr.) Leoben, 1902. 8vo. 12 pp., 1 plate.

- METER (W. F.). See ENCYKLOPÄDIE.
- MÜLLER (J. O.). Ueber die Minimaleigenschaft der Kugel. (Diss.) Göttingen, 1903. 8vo. 51 pp.
- PAMPUCH (A.). Das Malfatti-Steiner'sche Problem. (Progr.) Strassburg i. E., 1902. 4to. 73 pp., 10 plates.
- PASCAL (B.). Oeuvres complètes. Vol. 3: Traité divers de physique et de mathématiques. Table analytique. Paris, Hachette, 1903. 16mo. 502 pp. (Les principaux écrivains français.) Fr. 1.25
- POINCARÉ (H.). La science et l'hypothèse. Paris, Flammarion, 1903. 12mo. 284 pp. Fr. 3.50
- ROCQUIGNY-ADANSON (G. DE). Théorèmes sur les progressions arithmétiques; notes d'arithmologie; propositions de la théorie des nombres. Moulins, Auclair, 1903. 8vo. 20 pp.
- ROTHE. Lösung einiger Aufgaben über Flächenberechnungen mit Hilfe elliptischer Integrale. (Progr.) Nordhausen, 1902. 4to. 22 pp., 4 plates.
- SEYBOLD (C.). Die Drusenschrift: Kitāb Alnoqat Waldawāir, "Das Buch der Punkte und Kreise." Nach dem Münchener und Tübinger Codex. (Festschrift.) Tübingen, 1902. 4to. 111 pp.
- SEYLER (G.). Ueber die Erhaltung der Krümmungslinien bei Orthogonalprojektion. (Progr.) Passau, 1902. 8vo. 24 pp.
- SPENCER (F.). Ueber Konchoiden. (Progr.) Schwerin, 1902. 4to. 11 pp.
- STÖRMER (A.). See LIE (S.).
- TROPFKE (J.). Geschichte der Elementar-Mathematik in systematischer Darstellung. Vol. I: Rechnen und Algebra. Leipzig, 1902. 8vo. 8 + 332 pp. M. 8.00
- ZÄHLER (R.). Das Abelsche Theorem für Grundkurven, die in Gerade und Kegelschnitte zerfallen. (Diss.) München, 1902.

II. ELEMENTARY MATHEMATICS.

- AVERY (J. A.). Plane geometry by the suggestive method. Boston, Sanborn, [1903]. 12mo. 6 + 122 pp. Boards. \$0.40
- BACKHAUS (K.). See WIESE (B.).
- BAKER (W. M.) and BOURNE (A. A.). Elementary geometry, complete. London, Bell, 1903. 12mo. 506 pp. Cloth. 4s. 6d.
- BÉNÉZECH (L.). Cours d'algèbre. Livre I: Nombres algébriques (définition; opérations; applications). Rédigé conformément aux programmes officiels du 31 mai 1902, à l'usage des classes de troisième A et de quatrième B, des écoles primaires supérieures et professionnelles, et des candidats aux écoles nationales d'arts et métiers et aux écoles des mécaniciens des équipages de la flotte. Paris, Alcan, 1903. 18mo. 53 pp. Fr. 1.20
- BIEL (B.). Mathematische Aufgaben für die höheren Lehranstalten, unter möglichster Berücksichtigung der Anwendungen, wie überhaupt der Verknüpfung der Mathematik mit anderen Gebieten zusammengestellt. Ausgabe für Realanstalten. Teil I: Die Unterstufe. Leipzig, Freytag, 1903. 8vo. 6 + 206 pp. Cloth. M. 2.50

- BOURLET (C.). Cours abrégé d'arithmétique, contenant 1421 énoncés d'exercices. 3e édition, revue. Paris, Hachette, [1903]. 16mo. 414 pp. (Cours complet de mathématiques.) Fr. 2.50
- BOURNE (A. A.). See BAKER (W. M.).
- CRAWLEY (E. S.). Tables of logarithms, five-place; seven tables, with explanations. [New edition.] Philadelphia, Crawley, [1903]. 8vo. 32 + 76 pp. Cloth. \$0.75
- . The elements of trigonometry and five-place tables, bound together. [New edition.] Philadelphia, Crawley, [1903]. 8vo. Cloth. \$1.25
- DAVID (M. S.). Beginner's algebra. London, Black, 1903. 12mo. 2s. 6d.
- DEAKIN (R.). Euclid, books 5, 6, 11. London, Clive, 1903. 12mo. 152 pp. Cloth. (University tutorial series.) 1s. 6d.
- DICKSON (L. E.). Introduction to the theory of algebraic equations. New York, Wiley, 1903. 12mo. 5 + 104 pp. Cloth. \$1.25
- DUPRET (H.). Leçons d'algèbre, à l'usage des élèves de la quatrième B et de la troisième A de l'enseignement secondaire des lycées et collèges. Arras, Segaud, [1903]. 4to. 48 pp.
- FLETCHER (W. C.). See LACHLAN (R.).
- FREYCINET (C. DE). De l'expérience, en géométrie. Paris, Gauthier-Villars, 1903. 8vo. 19 + 178 pp. Fr. 4.00
- GIRNDT (M.). Raumlehre für Baugewerkschulen und verwandte gewerbliche Lehranstalten. Teil I: Lehre von den ebenen Figuren. Mit 207 der Baupraxis entlehnten Aufgaben. 2te Auflage. Leipzig, Teubner, 1903. 8vo. Cloth. M. 2.40
- GODFREY (C.). See HOCEVAR (F.).
- HOCEVAR (F.). Solid geometry; translated and adapted by C. Godfrey and E. A. Price. London and New York, Macmillan, 1903. 12mo. 6 + 80 pp. Cloth. \$0.50
- JIMÉNEZ RUEDA (C.). Programa de complementos de algebra y geometría. Madrid, Suárez, 1903. 8vo. 16 pp. (Universidad central.) Fr. 1.00
- KRAUS (K.). Grundriss der Geometrie und des geometrischen Zeichnens für Lehrerbildungsanstalten. Mit einem Situationsplan. Wien, Pichler, 1902. 8vo. 2 + 238 pp. Cloth. M. 2.70
- LACHLAN (R.) and FLETCHER (W. C.). Elements of geometry. London, Arnold, 1903. 12mo. 220 pp. Cloth. 2s. 6d.
- LICHTBLAU (W.). See WIESE (B.).
- LONDON matriculation model answers: mathematics. Papers in mathematics from January 1897 to September 1902. London, Clive, 1903. 12mo. 174 pp. (University tutorial series.) 2s.
- LUDWIG (F. K.). Das logarithmische Rechnen. Leichtfassliche Darlegungen über das Wesen, die Berechnung und Anwendung der Logarithmen nebst zahlreichen Beispielen und ausführlichen Lösungen. Zum Selbstunterricht bearbeitet. Reichenberg, Sollors, [1902]. 8vo. 3 + 52 pp. M. 1.00

NEUFFER. Elementare ebene Oerter. (Progr.) Ulm, 1902. 4to. 64 pp.

PETCH (T.). Plane geometry adapted to heuristic method of teaching. London, Arnold, 1903. 12mo. 120 pp. Cloth. 1s. 6d.

PRICE (E. A.). See HOCEVAR (F.).

SCHULTZ (E.). Kurzgefasstes Lehrbuch der Körperberechnung für gewerbliche Schulen. Essen, Baedeker, 1903. 8vo. 4 + 50 pp. Boards. M. 1.00

—. Nebst Aufgabensammlung. Essen, Baedeker, 1903. 8vo. 4 + 99 pp. Boards. M. 1.70

SIMÓN Y MAYORGA (J.). Programa de nociones de aritmética y geometría. Salamanca, Calón, 1902. 8vo. 18 pp.

WENTWORTH (G. A.). Logarithms, metric measures, and special subjects in advanced algebra. Boston, Ginn, 1903. 12mo. 141 pp. \$0.25

WIESE (B.), LICHTBLAU (W.), und BACKHAUS (K.). Raumlehre für Lehrerbildungsanstalten. (In 2 Teilen.) Teil 2: Stereometrie und Trigonometrie (Körperlehre und Dreiecksrechnung). 4te Auflage, umgearbeitet und erweitert nach dem preussischen Lehrplan für die Lehrerseminare vom 1. VIII. 1901 und in der neuen deutschen Rechtschreibung gedruckt. Breslau, Hirt, 1903. 8vo. 207 pp. M. 2.25

III. APPLIED MATHEMATICS.

ALIX (L.). Curso de geometría descriptiva, acompañado de un atlas dividido en tres partes. Valencia, Alufre, 1902. 4to. 112 + 199 + 75 pp., 31 + 48 + 13 plates. Fr. 20.00

ARRHENIUS (S. A.). Lehrbuch der kosmischen Physik. In 2 Teilen. Leipzig, Hirzel, 1903. 8vo. 8 + 8 + 1026 pp., 3 plates. M. 38.00

ARSONVAL (D'). La rotation de la terre démontrée par le pendule de Foucault. Appareil des écoles, approuvé par Flammarion. et Berget, présenté à l'Académie des sciences, le 17 novembre 1902. Les Châtelles, Raon-l'Étape, [1903]. 8vo. 32 pp., portrait.

BACH (C.). Die Maschinen-Elemente; ihre Berechnung und Konstruktion mit Rücksicht auf die neueren Versuche. 9te, vermehrte Auflage. 2 Bände. Stuttgart, Bergsträsser, 1903. 8vo. 22 + 848 pp., 59 plates. M. 32.00

BEHREND (B. A.). Le moteur d'induction. Traité succinct concernant sa théorie et son étude, avec de nombreux résultats d'essais et diagrammes. Paris, Loubat, 1903. 8vo. 2 + 117 pp.

—. See BERKITS (P.).

BENISCHKE (G.). Die Grundgesetze der Wechselstromtechnik. Braunschweig, Vieweg, 1903. 8vo. 10 + 141 pp. M. 3.60

BERKITS (P.). Induktionsmotoren. Ein Kompendium für Studierende und Ingenieure. Deutsche, autorisierte und erweiterte Bearbeitung von B. A. Behrend, The induction motor; unter Mitwirkung von W. Kübler herausgegeben. Berlin, Krayn, 1903. 8vo. 7 + 182 pp. M. 10.00

- BIRCKENSTAEDT (M.).** Verallgemeinerung der in den "Prinzipien der Mechanik für mehrere unabhängige Variable von Herrn L. Koenigsberger aus Heidelberg" dargestellten Hilfsätze über das kinetische Potential. (Diss.) Heidelberg, 1902. 4to. 51 pp.
- BIRVEN (H.).** Das Fachwerk. Eine Einführung in die statische Berechnung desselben; zugleich ein Repetitorium für den ausübenden Techniker. Hildburghausen, 1903. 8vo. 4 + 24 pp. Cloth. M. 1.50
- BODMER (G. R.).** Hydraulic motors and turbines; for the use of engineers, manufacturers, and students. New York, Van Nostrand, 1902. 8vo. Cloth. \$5.00
- BOLTZMANN (L.).** Ueber die Prinzipien der Mechanik. Zwei akademische Antrittsreden (1900 und 1902). Leipzig, Hirzel, 1903. 8vo. 48 pp. M. 1.00
- BRINK (A.).** Internal ballistics. Part I: Properties of powders and their action in closed vessels and in the bore of a gun. St. Petersburg, 1901-1902. 8vo. 12 + 370 pp. (Russian.) R. 3.40
- CAIN (W.).** Theory of steel-concrete arches and of vaulted structures; being a revised edition of "Voussoir arches applied to stone bridges, tunnels, culverts, and domes." New York, Van Nostrand, 1902. 16mo. Boards. (Van Nostrand's science series, No. 42.) \$0.50
- COMSTOCK (G. C.).** A text-book of field astronomy for engineers. New York, Wiley, 1902. 8vo. 10 + 202 pp. Cloth. \$2.50
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Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the American Mathematical Society, 501 West 116th Street, New York.

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501 West 116th Street, New York City.

The following dates have been fixed for the meetings of the Society

A. M.		A. M.	
Sat.,	Oct. 31, 1903, 11:00	Sat.,	Feb. 27, 1904, 11:00
M.-Tu.,	Dec. 28-29, ¹¹ ¹²	¹² April 24, ¹¹ ¹²	

The Tenth Summer Meeting and Fourth Colloquium of the Society will be held at the Massachusetts Institute of Technology, Boston, Mass., during the week beginning August 31, 1903.

J. L. L. J. L. L.

BULLETIN
OF THE
AMERICAN
MATHEMATICAL SOCIETY

CONTINUATION OF THE BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

F. N. COLE A. ZIWET D. E. SMITH F. MORLEY

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APR 3 1903

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 25, 1903, extending through the usual morning and afternoon sessions. About fifty persons were in attendance, including the following forty members of the Society.

Dr. Grace Andrews, Dr. C. L. Bouton, Professor E. W. Brown, Mr. J. S. Brown, Professor F. N. Cole, Miss E. B. Cowley, Professor E. S. Crawley, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Dr. William Findlay, Professor H. B. Fine, Professor T. S. Fiske, Miss Ida Griffiths, Professor G. H. Hallett, Miss Carrie Hamerslough, Professor James Harkness, Dr. H. E. Hawkes, Dr. J. I. Hutchinson, Dr. S. A. Joffe, Dr. Edward Kasner, Professor C. J. Keyser, Mr. L. L. Locke, Professor James Maclay, Dr. C. R. Mann, Professor E. H. Moore, Professor W. F. Osgood, Dr. I. E. Rabinovitch, Miss I. M. Schottenfels, Dr. Arthur Schultze, Dr. W. M. Strong, Mr. John Tatlock, Jr., Professor H. W. Tyler, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Mr. H. E. Webb, Professor A. G. Webster, Dr. J. K. Whittemore, Miss E. C. Williams, Dr. Ruth G. Wood, Professor R. S. Woodward.

The President of the Society, Professor Thomas S. Fiske, occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor W. N. Ferrin, Pacific University, Forest Grove, Ore.; Mr. E. H. Koch, Jr., Mackenzie School, Dobbs Ferry, N. Y.; Professor N. C. Riggs, Armour Institute of Technology, Chicago, Ill.; Mr. K. D. Swartzel, Harvard University, Cambridge, Mass. Thirteen applications for admission to the Society were received.

The following papers were read at this meeting:

- (1) DR. H. E. HAWKES: "On non-quaternion number systems in seven units."
- (2) PROFESSOR B. O. PEIRCE: "On families of curves which are the lines of certain plane vectors, either solenoidal or lamellar."
- (3) PROFESSOR E. W. BROWN: "On the variation of the arbitrary and given constants in dynamical equations."

(4) Dr. L. P. EISENHART: "Congruences of tangents to a surface, and derived congruences."

(5) Dr. H. F. STECKER: "Least distance in the non-euclidean plane."

(6) Professor L. E. DICKSON: "Fields whose elements are linear differential equations."

(7) Dr. SAUL EPSTEIN: "On linear differential congruences."

(8) Professor R. S. WOODWARD: "The deviation from the vertical of falling bodies" (preliminary communication).

(9) Dr. EDWARD KASNER: "The automorphic groups of the manifolds defined by a general and a symmetric determinant."

(10) Mr. C. H. SISAM: "On some directrix curves on quintic scrolls."

(11) Mr. L. I. NEIKIRK: "Groups of order p^n which contain cyclic subgroups of order p^{n-3} ."

(12) Miss I. M. SCHOTTENFELS: "On the simple groups of order $8!/2$."

(13) Dr. E. B. WILSON: "The so-called foundations of geometry."

Mr. Sisam's paper was communicated to the Society through Dr. Snyder; Dr. Epstein's through Professor Dickson. Mr. Neikirk was introduced by Professor Crawley. In the absence of the authors, Mr. Sisam's paper was read by Dr. Hutchinson, and the papers of Professor Peirce, Dr. Stecker, Professor Dickson, Dr. Epstein and Dr. Wilson were read by title.

Professor Dickson's paper will be published in the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. At the December meeting of the Society Dr. Hawkes gave a method for the enumeration of all non-quaternion number systems with skew units. The present paper applies this method to the case $n = 7$ and deduces expeditious means of carrying out further enumerations should it be desirable.

2. If $u \equiv f(x, y) = c$ represents any family of curves in the xy plane, any vector V the components of which are of the form

$$\left(F(x, y), -F(x, y) \cdot \frac{\partial u}{\partial x} / \frac{\partial u}{\partial y}, 0 \right)$$

where F is arbitrary, evidently has the u curves for its lines. Of all the vectors which have these given lines, an infinite number are in general solenoidal and an infinite number are lamellar, but no one can be both solenoidal and lamellar, unless u happens to satisfy Lamé's well known condition for an isothermal parameter $[\nabla^2(u)/h_u^2$ a function of u only], in which case there are a set of such vectors connected by a linear relation.

In certain general problems in mathematical physics, classes of lamellar or solenoidal vectors appear which are known to satisfy in each case some general condition common to all of the class. Sometimes, for instance, the tensor, or the divergence, or the tensor of the curl of a vector must be constant in a given region; or the tensor, or the divergence, or the curl must be expressible in terms of the parameter of the lines of the vector, or in terms of the parameter of the orthogonal lines. Professor Peirce's paper discusses briefly the possible lines of plane vectors which satisfy a few of the more common conditions. For example, it is shown that, if $u = c$, $v = k$ are the equations of curves in the xy plane which cut each other orthogonally, the condition that the u curves shall be possible lines of a set of solenoidal vectors, the tensors of which involve u only, is the condition that the v curves shall be possible lines of a set of lamellar vectors, the tensors of which involve u only. If the u curves are possible lines of a solenoidal vector, the tensor of which is expressible in terms of v , the v curves are the lines of a lamellar vector, the tensor of which is a function of v only. If a solenoidal vector which has the u curves for lines has a tensor expressible in terms of u , the u curves must form a parallel system, etc., etc.

3. Professor Brown's paper contains certain results with reference to the arbitrary constants which arise in the integration of a set of dynamical equations and also with reference to the constants which occur in the force function. The equations are supposed to have been solved by trigonometric series with arguments some of which are arbitrary and some of which are present in the force function. In the latter part of the paper it is considered under what circumstances we may substitute a variable value for a constant as well in the solution as in the force function. It then appears that Newcomb's well known theorem on the secular accelerations is a particular case of sub-

stitution of a variable for a constant in the solution of a set of such differential equations.

4. Given a family of curves upon a surface S ; the tangents to these curves form a rectilinear congruence for which S is one of the focal sheets. The other focal sheet is a determinate surface S_1 , and upon it there is a family of curves to which the lines of the congruence are tangent. If tangents are drawn to the curves on S_1 , which are conjugate to the latter, a second congruence is formed with S_1 for one focal sheet and another surface S_2 for the other sheet. This process can be continued indefinitely unless one of the surfaces reduces to a curve. In a similar manner we can get another sequence of congruences by starting with the tangents to the conjugates of the given curves on S . These are the *derived congruences* of Darboux. After finding the expressions for the different functions of these congruences in terms of functions of S , Dr. Eisenhart applies them to a study of such sequences where all the congruences of the series are of a particular form. Thus it is shown that the system of curves upon three consecutive surfaces cannot be formed of lines of curvature, so that there cannot be a sequence of congruences of Guichard, as Bianchi calls those whose developables touch the focal sheets in lines of curvature of the latter. From this it follows almost immediately that there are no series of congruences of which each member is a normal congruence.

If a congruence of Ribaucour is defined as one whose developables cut the mean surface in a conjugate system, the condition that the tangents to the curves $v = \text{const.}$ of a surface form such a congruence takes a very simple form. From this it is found that, when the tangents to the curves $u = \text{const.}$ conjugate to the former also form a congruence of this kind, all the members of the sequence are congruences of Ribaucour. A large number of surfaces are found to have a conjugate system which has this property. It is shown that the only isothermic surfaces whose lines of curvature form such a system have either of the following linear elements when referred to this system :

$$ds^2 = UV(du^2 + dv^2), \quad ds^2 = e^{UV}(du^2 + dv^2),$$

where U is a function of u alone and V is a function of v alone.

When one proceeds to the determination of the surfaces with

these linear elements, he finds that U or V must be constant, so that the only surfaces are surfaces of revolution. From this one is led to the result that the tangents to the meridians of a surface of revolution form a normal congruence of Ribaucour.

The equations of condition that the tangents to a given family of curves on S form a cyclic congruence can be brought to a very simple form, so that quite a number of theorems follow almost immediately. Among others one notes that when the tangents to the lines of curvature in one system upon a surface form a cyclic congruence, it is at the same time a congruence of Ribaucour and conversely. The discussion closes with an inquiry as to the existence of a sequence of cyclic congruences for which the circles of each congruence are equal, and it is found that there are none of this kind.

5. The problem treated in Dr. Stecker's paper is an application of the calculus of variations to determine, directly from the distance integral, the least distance between two fixed points in the non-euclidean plane.

7. In the *Comptes Rendus*, 1897, page 489, Guldberg shows that it is possible to find for linear differential forms a theory which is analogous to the Galois field theory. In the present paper Dr. Epstein develops this theory up to the point of demonstrating the theorem: Every existent differential field of order s may be represented as a Guldberg field of order $s = p^n$; the $GuF(p^n)$ is defined uniquely by its order; in particular, it is independent of the special irreducible differential congruence used in its construction. This is the exact analogue of a well-known theorem of E. H. Moore (Chicago Congress Papers, page 220; Dickson, Linear Groups, page 14).

8. In his preliminary communication Professor R. S. Woodward reviewed the history of the problem of the deviation from the vertical of falling bodies from the time of its earliest treatment by Gauss * and Laplace,† in 1803, down to the present time. He also presented a solution of the problem differing from any of the solutions hitherto published, but giving the same results, to terms of the order of the square of the angular velocity of the earth, as those derived by Gauss, and

* *Werke*, vol. 5. pp. 495-503.

† *Mécanique Céleste*, vol. 4, chapter 5.

as those required by, though not fully worked out in, Laplace's investigation.

9. The first part of Dr. Kasner's paper is devoted to certain general principles of invariant theory, while the second is an application to the problem announced in the title. If the coefficients of a form are interpreted as homogeneous point coördinates in a space of sufficiently high dimension, then any invariant equated to zero represents a variety which is transformed into itself by a certain group of collineations. This group includes the so-called induced group of the coefficients, and in many cases coincides with it. It follows that any covariant (in the domain of the coefficients) of an invariant is also an invariant, a principle which includes for example the Aronhold process as a very special case.

The conclusion concerning the general determinant

$$\Delta = |a_{ik}| = 0$$

is that its collineation group is composed of two continuous systems containing $2(n^2 - 1)$ parameters. The proof involves, in addition to the general principles, a geometric discussion of the manifold $\Delta = 0$ in space of $n^2 - 1$ dimension, in particular its two-fold projective generation. The case $n = 2$ gives of course the well known six parameter group connected with a quadric surface.

10. Mr. Sisam's paper proved the two following theorems :
 1° Every unicursal quintic scroll has three coplanar generators ; the residual may be either a conic, a double directrix or a simple directrix and a fourth generator. 2° If the asymptotic lines of a (4, 1) scroll are reducible, two real and two imaginary generators issue from each real point of the four-fold line.

11. The groups of operations of order p^m which contain a cyclic subgroup of order p^{m-2} have been determined by Miller (see *Transactions*, volume 3, number 4). In the present paper Mr. Neikirk has determined the groups of order p^m which contain a cyclic subgroup of order p^{m-3} . The investigations are number-theoretic in character and are based on a division of the groups considered into classes. This division is shown to depend on the partitions of m which give the orders of the independent operations that enter into the con-

struction of the required group. Denoting the cyclic subgroup of order p^{m-3} by $\{P\}$ and the partition by $[a, b, \dots]$, the numbers of resulting groups are given by: $[m-3, 3]$, 12 groups, 4 in which $\{P\}$ is self-conjugate, including the abelian group of this type, and 8 in which $\{P\}$ is not self-conjugate; $[m-3, 2, 1]$, $20+p$ groups in $8+p$ of which $\{P\}$ is self-conjugate; $[m-3, 1, 1, 1]$, 5 groups, in all of which $\{P\}$ is self-conjugate. The total number of these groups is therefore $87+p$.

The formal generational equations of all these types in terms of independent operators are given in the course of the investigation.

12. Miss Schottenfels's paper contains the proof that: 1° All types of simple groups of order $8! / 2$ or 20160, contain at least 960 conjugate subgroups of order 7. 2° No simple group of this order contains an operator whose period is a multiple of 7 greater than 1.7, and all such simple groups are identical, with respect to the number of such operators or elements.

13. The first part of Dr. Wilson's paper contains criticisms on Professor Hilbert's "Grundlagen der Geometrie" which appeared in the *Mathematische Annalen*, October, 1902. The second part is occupied with the construction of some Hilbertian geometries and in particular a two-dimensional geometry in three dimensions. The paper is to appear in the *Archiv der Mathematik und Physik*.

F. N. COLE.

COLUMBIA UNIVERSITY.

THE APRIL MEETING OF THE CHICAGO SECTION.

A regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held on Saturday, April 11, 1903, at the Armour Institute of Technology, in Chicago, opening at 9 A. M. The following members were present:

Professor W. W. Beman, Professor D. F. Campbell, Dr. J. V. Collins, Professor E. W. Davis, Professor B. F. Finkel, Mr. W. B. Ford, Professor Thomas F. Holgate, Professor A. M. Kenyon, Professor Kurt Laves, Dr. A. C. Lunn, Professor H. Maschke, Professor Malcolm McNeill, Professor E. H. Moore, Professor F. R. Moulton, Professor E. B. Skinner, Professor C. A. Waldo, Professor H. S. White, Professor B. F. Yanney, Professor J. W. A. Young, Professor A. Ziwet.

Professor Waldo was elected chairman. The following papers were read:

(1) Mr. OSWALD VEBLEN: "A set of independent axioms of geometry."

(2) Dr. F. R. MOULTON: "The conditions for the convergence of the expressions for the ratios of the triangles when developed as power series in the time intervals."

(3) Professor A. S. CHESSIN: "On the strains in a rapidly rotating disc."

(4) Dr. SAUL EPSTEEN: "On Loewy's fundamental theorem."

(5) Professor PETER FIELD: "On the form of a plane quintic curve with five cusps."

(6) Professor H. S. WHITE: "Polar triangles of a conic and triply tangent conics of a cubic."

(7) Professor H. MASCHKE: "Invariants and covariants of quadratic differential quantics in n variables."

(8) Mr. E. L. HANCOCK: "Asymptotic lines on certain surfaces of revolution."

(9) Professor ARCHIBALD HENDERSON: "On the construction of a double-six."

(10) Professor ARCHIBALD HENDERSON: "On the graphic representation of the straight lines upon the twenty-one different types of the cubic surface."

(11) Mr. O. J. FERGUSON: "Problems in projective geometry."

(12) Professor E. W. DAVIS: "Some groups in the logic of relatives."

(13) Mr. N. J. LENNES: "Theorems on the polygon and the polyhedron."

Mr. Veblen and Mr. Lennes were introduced to the Section by Professor Moore, and Professor Henderson by Professor Maschke; Mr. Ferguson's paper was presented through Professor Davis. In the absence of Professor Chessin and Mr. Hancock their papers were read by title.

The report of the committee on the requirements for the Master's degree, with mathematics as the major subject, was taken up for consideration and after discussion was again laid on the table till the next meeting.

The North Central Association of Science Teachers being in session at Armour Institute, a joint meeting was arranged with the mathematics section of this association, at which Professor W. W. Beman presided and the following papers were read:

"Introduction of physical and other problems into elementary mathematical courses, and the necessary consequences," by Professor C. E. COMSTOCK, of Bradley Polytechnic Institute.

"What is the laboratory method?" by Professor J. W. A. YOUNG, of the University of Chicago.

The discussion of these papers was opened by Miss Long, of the Lincoln, Nebraska, High School.

Abstracts of the papers presented at the Sectional meeting are as follows:

1. The aim of Mr. Veblen's paper, which is intended as a dissertation for the doctor's degree at the University of Chicago, is to present a system of axioms that are at once independent of one another and sufficient for the deduction of the euclidean geometry. The axioms are stated in terms of two fundamental symbols, point and order, and thus follow the trend of development inaugurated by Pasch and continued by Peano, rather than that of Hilbert. All other geometrical concepts, such as line, plane, space, motion, are defined in terms of point and order. In particular, the congruence relations are made the subject of definitions rather than of axioms. This is accomplished by the aid of projective geometry according to the method first given analytically by Cayley and Klein.

2. In rectangular heliocentric coördinates the functions considered in Professor Moulton's paper are

$$(1) \quad \frac{x_i y_j - y_i x_j}{y_k y_l - y_k x_l},$$

where the x and y are defined by the differential equations

$$(2) \quad \frac{d^2 x}{dt^2} = -\frac{k^2 x}{r^3}, \quad \frac{d^2 y}{dt^2} = -\frac{k^2 y}{r^3}.$$

All the singularities of x and y considered as functions of t are found and also the conditions under which the denominator of (1) can vanish. Having located all the singularities of the function, the true radius of convergence around any initial point is easily found. The problem is solved in all detail for all classes of conics, and tables are constructed giving numerical results for the various cases which can arise. The relation of the limits in this part of the theory of orbits, to those of the series used in the *Astronomical Journal*, No. 510, is given, and an exhibition of the corresponding question in Laplace's method. The paper will be published in the *Astronomical Journal*.

3. Professor Chessin finds expressions for the displacements of a particle in a thin rotating disc in terms of the coördinates of the particle and certain constants determined from the surface conditions.

5. Del Pezzo has shown how it is possible to obtain the equation of a five cusped quintic curve by inverting a quartic which has two cusps and which is inscribed in and circumscribed about the fundamental triangle. The equation of the quintic is written in the form $C_4^2 - \phi^3 \phi_1 = 0$, where C_4 is a quartic and the ϕ 's are conics; the constants being so chosen that x, y, z are factors of the equation. Professor Field divides the plane into regions by drawing the curves C_4, ϕ, ϕ_1 , and by their aid obtains the form of the curves represented by the above equation.

6. A plane cubic may be so situated with reference to a conic as to admit an inscribed triangle, self-conjugate with respect to the conic, and having one degree of freedom. It is shown by

Professor White that the vertices of such a triangle will be contacts of a tritangential conic. There are 3 systems of such conics, corresponding to the 3 pro-Hessians of the cubic. The tritangential conics of a simply infinite system like that mentioned above are poloconics of a simple infinity of straight lines, with respect to some one of the 3 pro-Hessians. It is found that these lines form a pencil, all passing through one common point. There is thus shown the existence, under special conditions, of a covariant point in the system of a conic and a cubic.

7. Professor Maschke's paper will be published in the *Transactions*.

8. Mr. Hancock considers two surfaces of revolutions S and S_1 , generated by the curves C and C_1 . C_1 is formed by taking on the tangents to C a length equal to t^{-1} times the length of the tangent. The generating curves then are

$$z = f(u), \quad z_1 = (l-1) u_1 f'(lu_1) + f(lu_1),$$

and the equation of transformation from S to S_1 , $u = lu_1$, $v = v_1$.

The condition that the asymptotic lines on S go over into the asymptotic lines on S_1 by the transformation is $D_1/D_1'' = e^2 D/D''$. This is seen in particular when the generating curves are $z = u^{**} + C_2$ and $z_1 = k_1 u_1^{**} + C_2$. The paper will be offered to the *American Journal of Mathematics* for publication.

9. A double-six, in the Schläfli-Cayley notation, is represented by the scheme

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1' & 2' & 3' & 4' & 5' & 6' \end{pmatrix}$$

and is defined to be two systems of six lines each, such that no two lines of the same system intersect, but that each line of the one system intersects all but the corresponding line of the other system. In Professor Henderson's paper there is given a simple method of constructing a double-six, viz., if we select the equations of the five lines $1'$, $3'$, $4'$, $5'$, $6'$, such that they do not intersect, but such that they are all met by the same straight line 2 , whose equations are also known, then the equations of the remaining six lines 1 , 3 , 4 , 5 , 6 and $2'$, are uniquely deter-

mined. Using quadriplanar coördinates and properly choosing the constants, a thread model of the configuration was very easily constructed.

10. Cayley in his Memoir on cubic surfaces (Collected mathematical papers, volume VI (1868), pages 359–455), has given a treatment of the twenty-three types of the cubic surface, the division depending upon the nature of the singularities, and not upon the reality of the lines. The equations there given are canonical forms in quadriplanar coördinates. Two of the surfaces however are scrolls, in which there is no question of the twenty-seven lines.

In Professor Henderson's second paper there is developed a simple method of graphically representing the lines on the remaining twenty-one types of the cubic surface, given by the above-mentioned canonical forms. By this method the mutual relations of the lines, as well as their relation to the fundamental tetrahedron, are shown and the results obtained are equally adapted to the construction of thread models or perspective drawings of the various configurations.

The paper was illustrated by perspective drawings in colors of the lines upon the twenty-one types of the cubic surface, and incidentally there was shown a graphic representation of the projection of the lines upon a cubic surface with a conical point into the Pascal configuration.

11. Mr. Ferguson's paper contained demonstrations of the following theorems : 1° If a range of points μ is projective to a pencil of rays S , and if through the points of μ there are drawn rays perpendicular to the corresponding rays of S , these generate a ruled surface of the fourth order. 2° In two projective sheaves of rays the common perpendiculars of corresponding rays generate a ruled surface of the sixth order.

12. Professor Davis established a one to one correspondence between the different ways in which $3, 4, \dots, n$ relatives can be connected by logical multiplication and addition, and the substitutions on $3, 4, \dots, n$ letters.

13. Under formal definitions of polygon and polyhedron, Mr. Lennes proved certain theorems on the partition of the plane and space.

THOMAS F. HOLGATE,
Secretary of the Section.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION.

THE third regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held on Saturday, April 25, 1903, at Stanford University. The following fifteen members were present :

Professor R. E. Allardice, Dr. E. M. Blake, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor R. L. Green, Professor L. M. Hoskins, Dr. D. N. Lehmer, Dr. J. H. McDonald, Professor G. A. Miller, Dr. H. C. Moreno, Dr. T. M. Putnam, Professor Irving Stringham, Mr. L. C. Walker, Mr. A. W. Whitney, Professor E. J. Wilczynski.

A morning and an afternoon session were held, Professor Stringham acting as chairman at both sessions. Professor M. W. Haskell was elected a member of the programme committee in place of Professor E. J. Wilczynski, who will be abroad next year.

The following papers were read at this meeting :

(1) Professor E. J. WILCZYNSKI: "Invariants of systems of linear partial differential equations, and the theory of congruences."

(2) Dr. D. N. LEHMER: "Preliminary report on a table of smallest divisors."

(3) Professor H. F. BLICHFELDT: "Note on linear substitution groups of finite order."

(4) Professor R. E. ALLARDICE: "On some curves connected with a system of similar conics through three points."

(5) Dr. SAUL EPSTEIN: "Necessary and sufficient condition for the existence of invariant subgroups."

(6) Professor G. A. MILLER: "On reciprocal groups."

(7) Dr. H. C. MORENO and Professor G. A. MILLER: "On the non-abelian groups in which every subgroup is abelian."

(8) Mr. W. A. MANNING: "On the class of primitive substitution groups."

(9) Miss Ida M. SCHOTTENFELS: "Generational definition of an abstract group simply isomorphic with the simple substitution group G_{20160}^{21} ."

(10) Dr. T. M. PUTNAM: "Certain subgroups of the quater-

nary linear fractional group of determinant unity, in the general Galois field."

The papers by Dr. Epstein and Miss Schottenfels were read by Professor Wilczynski and the secretary respectively. All the other papers were read by their authors. Abstracts are given below.

1. Professor Wilczynski considers in his paper a system of two linear homogeneous partial differential equations with two independent variables x_1 and x_2

$$(1) \quad \begin{aligned} \Omega &= ay_1 + by_2 + cz_1 + dz_2 + ey + fz = 0, \\ \Omega' &= a'y_1 + b'y_2 + c'z_1 + d'z_2 + e'y + f'z = 0, \end{aligned}$$

where $y_k = \partial y / \partial x_k$, $z_k = \partial z / \partial x_k$, ($k = 1, 2$).

If one makes the transformations

$$(2) \quad y = a\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta, \quad (a\delta - \beta\gamma \neq 0);$$

$$(3) \quad \xi_1 = f_1(x_1, x_2) \quad \xi_2 = f_2(x_1, x_2), \quad \left(\frac{\partial(\xi_1, \xi_2)}{\partial(x_1, x_2)} \neq 0 \right),$$

where $a, \beta, \gamma, \delta, f_1$ and f_2 are arbitrary functions of x_1 and x_2 , (1) is transformed into another system of the same form. Moreover one may replace the system by

$$(4) \quad \phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0, \quad \phi\omega - \psi\chi \neq 0,$$

where ϕ, ψ, χ, ω are arbitrary functions of x_1 and x_2 , and obtain again a system of the same form. These are the most general transformations which convert a general system (1) into another of the same form.

The author considers the functions of the coefficients of (1) which remain invariant under these transformations, and actually computes them for a reduced form.

If one considers four sets of simultaneous solutions

$$y^{(k)} = g^{(k)}(x_1, x_2) \quad z^{(k)} = h^{(k)}(x_1, x_2), \quad (k = 1, 2, 3, 4)$$

one may interpret them as the homogeneous coordinates of two points in space, the locus of each being a surface. The transformations (2), (3) and (4) merely displace these points on the line joining them and otherwise change merely the parametric

representation of the two surfaces. They leave invariant, therefore, the congruence of lines obtained by joining corresponding points of the two surfaces. The theory of a system of form (1) is therefore geometrically a theory of congruences. This leads to geometric interpretations for the vanishing of certain invariants, the determination of the developable surfaces of the congruence, its focal surfaces, etc. The properties thus obtained are all projective. But not all projective properties will appear, owing to the fact that congruences which are not mere projective transformations of each other, may belong to the same system (1). This question is investigated in the paper. All systems of form (1) may be reduced to a few simple types. The general type corresponds to a congruence with distinct non-degenerate focal surfaces. For instance, the system of partial differential equations of the theory of functions belongs to a congruence whose focal surfaces have degenerated into two distinct curves, and may be taken as the type of such systems.

The theory of the Laplace transformations of an equation of the form

$$\frac{\partial^2 \theta}{\partial x \partial y} + a \frac{\partial \theta}{\partial x} + b \frac{\partial \theta}{\partial y} + c \theta = 0,$$

which is so important in the theory of surfaces, is also generalized to a system of the kind considered and finds its geometrical interpretation the same as in that older theory. It is of course connected with the theory of the covariants of such a system.

The author points out moreover that this theory can be generalized for other systems, and that geometrically other configurations, complexes, etc., may be studied in this way.

2. Dr. Lehmer's factor tables are intended to give the smallest divisors of all numbers less than ten millions. Multiples of 2, 3, 5 and 7 are not tabulated. All other numbers are of the form $210x + R$ where R is prime to 210. There are 48 values of R which give the headings for as many columns. The values of x are listed at the side. Each page of the tables contains 100 rows and 48 columns and serves to factor 21,000 numbers. The plan of the table is somewhat similar to that used by Lebesgue (*Tables diverses pour la décomposition des nombres en leurs facteurs premiers*. Par V. A. Lebesgue, Paris, 1864), but differs from it in several important particulars.

3. In determining the linear homogeneous substitution groups of finite order in two variables, Gordan makes use of the equation $1 + \cos \phi_1 + \cos \phi_2 + \cos \phi_3 = 0$; ϕ_1, ϕ_2, ϕ_3 being rational angles ("Ueber endliche Gruppen linearer Transformationen einer Veränderlichen," *Mathematische Annalen*, 1877, page 29). By forming the corresponding equation for a linear homogeneous group G in n variables it may be proved that all the substitutions of such a group whose orders are products of primes greater than $(n-1)(2n+1)$ form a subgroup G_1 . Professor Blichfeldt proved that G_1 is abelian and self-conjugate under G . He also pointed out the following results: (1) No simple group whose order is divisible by a prime greater than $(n-1)(2n+1)$ can be simply isomorphic with a linear homogeneous substitution group in n variables. (2) If the determinants of the substitution of a linear homogeneous substitution group in n variables are all unity, and if n is a prime, then the order of the group is not divisible by any prime greater than $(n-1)(2n+1)$.

4. This paper may be considered a continuation of the one published in the *Transactions*, volume 4 (1903), page 103. In it Professor Allardice gives an investigation of the loci of the vertices and foci of the system of conics through three points. He also considers the envelope of the directrices. Each pair of vertices and each pair of foci has a distinct locus, and each directrix has a distinct envelope. The loci are of the eighth and sixth degrees respectively, and the envelope is of the fourth degree.

5. Dr. Epstein shows that a certain theorem of Lie's may be stated as follows: The necessary and sufficient conditions that a continuous r -parameter group G_r may contain an invariant subgroup g_s ($s < r$), is that the adjoint group G_r shall be reducible. A number of consequences are easily deduced from this by utilizing the notions of reducibility of the adjoint group. For instance, if G_r is integrable, so is its adjoint group. If a group coincides with its adjoint, and contains an invariant subgroup, it is reducible. The quotient groups are the same for all composition series apart from their order. By introducing a notion, which in the theory of continuous group corresponds to the group of contragredient isomorphisms, the "total adjoint group" is defined, and the total adjoint group is then shown to be always reducible.

6. Several years ago Professor Tanner called attention * to the indications of the existence of a reciprocity theorem, viz., that in every group of totitives of order g there are as many subgroups of order g_1 as there are of order $g + g_1$, g_1 being any divisor of g . As the groups of totitives are included among the abelian groups,† the theorem whose existence Professor Tanner suspected is included in the theorem that every abelian group of order g has as many distinct subgroups of order g_1 as there are subgroups of order $g + g_1$. The last theorem is readily proved by means of the characteristics of an abelian group. Professor Miller proved this theorem without using characteristics and pointed out some of its applications.

7. At the first meeting of the Section, Dr. Moreno presented some results in regard to non-abelian groups in which every subgroup is abelian.‡ The present paper completes the determination of all such groups. The following are some of the most important results: The order of such a group cannot be divided by more than two distinct primes. If it is divisible by two primes (p, q) , the Sylow subgroup whose order is a power of one of these primes is of type $(1, 1, 1, \dots)$, while the other Sylow subgroup is cyclic. The former of these Sylow subgroups is also the commutator subgroup and includes no invariant operator besides the identity. When the order of such a group is p^a , it contains just $p + 1$ subgroups of order p^{a-1} and none of these can have more than three invariants. The paper will be offered to the *Transactions* for publication.

8. Mr. Manning investigated the primitive substitution groups of class $3p$, p being a prime, which contain substitutions of order p and degree $3p$. Jordan announced that the degree of a primitive group containing such a substitution cannot exceed $3p + 4$. By using this theorem it is shown that such groups do not exist when $p > 5$. When $p = 5$, there are the three primitive groups, G_{80}^{16} , G_{240}^{16} , and G_{4080}^{17} . The two cases, in which $p = 2$ and $p = 3$, are not here considered since they have already been worked out by Jordan.

* Tanner, *Proc. London. Math. Soc.*, vol. 27 (1896), p. 329.

† *Annals of Math.*, vol. 2 (1900), p. 79.

‡ BULLETIN, vol. 8 (1902), p. 434. When P_{n-1} has only two invariants each of them may be unity. This special case was overlooked.

9. In a paper presented to the Society, October, 1901, Miss Schottenfels determined the generational definition of the abstract group of order 960 holodrically isomorphic to the ternary group Galois field $[2^2]$ of the same order with the following definition :

$$(1) \quad \begin{aligned} E_1^3 &= E_2^3 = E_3^3 = B_1^3 = I, \\ (E_1 E_2)^3 &= (E_2 E_3)^3 = (E_1 E_3)^3 = I, \\ (B_1 E_1) &= (B_1 E_2)^4 = (B_1 E_3)^3 = I, \end{aligned}$$

where E_1, E_2, E_3, B_1 satisfying relations (1) are the generators of the group.

This group can be immediately extended by an operator C of order 7, say $C^7 = I$, to the abstract simple group of order $8! / 2$ holodrically isomorphic to the simple ternary group Galois field $[2^2]$ of order $8! / 2$, with generators C, E_1, E_2, E_3, B_1 , satisfying the further relations

$$(2) \quad (CE_1)^5 = (CE_2)^5 = (CE_3)^5 = (CB_1)^5 = I.$$

10. In the determination of the cyclic subgroups of the quaternary group of determinant unity, taken in a general Galois field, the canonical forms with unequal multipliers give rise to some difficulties which do not present themselves in the others. Dr. Putnam investigates these cases, starting with the homogeneous group and passing from it to the fractional form in the usual way. The paper is supplementary to the article published in the *American Journal of Mathematics*, volume 24 (1902), page 319.

G. A. MILLER,
Secretary of the Section.

STANFORD UNIVERSITY.

A FUNDAMENTAL THEOREM WITH RESPECT TO TRANSITIVE SUBSTITUTION GROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, February 28, 1903.)

LET G represent any transitive group of degree n and of order $g = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$; $p_1, p_2, \dots, p_m$ being distinct primes. According to Sylow's theorem G contains at least one subgroup of each of the orders $p_1^{a_1}, p_2^{a_2}, \dots, p_m^{a_m}$. It will be convenient to speak of these subgroups as the *Sylow subgroups* of G . Since G is transitive, all the prime factors of n are included among the factors of g . The theorem in question may now be stated as follows:

THEOREM. *If p_1^β is the highest power of p_1 which divides n , each Sylow subgroup of order $p_1^{a_1}$ in G has a transitive constituent of degree p_1^β and all its other transitive constituents are of degree $p_1^{\beta+\gamma}$, $\gamma \equiv 0$.*

COROLLARY I. *If $n = 2p^a$, p being any odd prime, each of the Sylow subgroups whose order is a power of p has just two transitive constituents, each being of degree p^a .*

COROLLARY II. *If n is a power of a prime, a Sylow subgroup of G whose order is a power of the same prime must be transitive.*

The proof of the theorem results from the following elementary considerations: Let G_1 represent the subgroup of G which is composed of all its substitutions which omit a given letter. Every Sylow subgroup of G_1 is found in some Sylow subgroup of G . Whenever the orders of these subgroups are different the former is composed of all the substitutions of the latter which omit one letter.*

Since p_1 is any prime divisor of g we may confine our attention to the Sylow subgroups of G_1 and G whose orders are $p_1^{a_1-\beta}$ and $p_1^{a_1}$ respectively. At least one transitive constituent of the latter must be such that the order of its largest subgroups of lower degree than its own may be obtained by dividing its own order by p_1^β ; i. e., one of its transitive consti-

* Cf. Burnside, Theory of groups of finite order, 1897, p. 94.

tients must be of degree p_1^β . If another transitive constituent were of degree p_1^δ , $\delta < \beta$, G_1 would contain a subgroup of order $p_1^{\alpha-\delta} > p_1^{\alpha-\beta}$. As this is impossible, a Sylow subgroup of order p_1^α contains at least one constituent of degree p_1^β but it contains no constituent of lower degree.

This proof remains true even if $\beta = 0$, for in this case a Sylow subgroup of order p_1^α is of degree less than n and hence may be supposed to have a constituent of degree 1. In all other cases the degree of this Sylow group is evidently equal to n . That the Sylow subgroup of order p_1^α may have constituents whose degrees exceed p_1^β follows directly from the three transitive groups of degree 6 and order 24. In two of these the Sylow subgroups of order 8 involve just two transitive constituents, of degrees 4, 2 respectively, while our theorem merely proves that there is a constituent of degree 2 and that there cannot be one of a lower degree.

As an application of this theorem we may prove the following: *If the degree of a simply transitive primitive group is np^a , p being any prime which does not divide $n > 1$, the maximal subgroup of degree $np^a - 1$ cannot involve a transitive constituent of degree $(n-1)p^a$. If $a = 0$ this theorem requires no proof. If $a > 0$, the Sylow subgroup of order p^a in G would contain two transitive constituents, of degrees $(n-1)p^a$, p^a , respectively. Hence G would involve a substitution not found in G_1 , which would not connect all the transitive constituents of G_1 . This is impossible since G_1 is maximal in a primitive group. For instance, in a simply transitive primitive group of degree 18, the maximal subgroup of degree 17 cannot involve a transitive constituent of degree 9.*

STANFORD UNIVERSITY,
February, 1903.

THE CHARACTERIZATION OF COLLINEATIONS.

BY DR. EDWARD KASNER.

(Read before the American Mathematical Society, February 28, 1903.)

A COLLINEATION is ordinarily defined as a point transformation which converts collinear points into collinear points, *i. e.*, one for which the family of ∞^2 straight lines, or the equivalent differential equation $y'' = 0$, is invariant. The question suggests itself whether this definition does not contain redundancies—whether it is not sufficient to require that only some straight lines shall remain straight. If we understand by a *simple* system of lines any system possessing the property that through each point of the region of the plane considered there passes one and only one line, then the result of this note may be stated :

If four simple systems of straight lines remain straight after a point transformation, then the same is necessarily true of all straight lines, and the transformation is, therefore, a collineation.

To prove this consider the general point transformation T

$$(1) \quad X = \phi(x, y), \quad Y = \psi(x, y),$$

where ϕ and ψ are one-valued continuous functions possessing first and second derivatives in the region considered. It is assumed, of course, that the Jacobian, $J = \phi_x \psi_y - \psi_x \phi_y$, does not vanish identically. The once extended transformation is

$$(2) \quad Y' = \frac{\psi_x + \psi_y y'}{\phi_x + \phi_y y'};$$

and the twice extended may be written *

$$(3) \quad Y'' = \frac{a + \beta y' + \gamma y'^2 + \delta y'^3 + J y''}{(\phi_x + \phi_y y')^3},$$

where

$$(4) \quad \begin{aligned} a &= \phi_x \psi_{xx} - \psi_x \phi_{xx}, \\ \beta &= \phi_y \psi_{xx} - \psi_y \phi_{xx} + 2\phi_x \psi_{xy} - 2\psi_x \phi_{xy}, \\ \gamma &= \phi_x \psi_{yy} - \psi_x \phi_{yy} + 2\phi_y \psi_{xy} - 2\psi_y \phi_{xy}, \\ \delta &= \phi_y \psi_{yy} - \psi_y \phi_{yy}. \end{aligned}$$

* Lie-Scheffers, *Continuirliche Gruppen*, 1893, p. 33.

If now T transforms any straight line into a straight line, then for that line $y'' = 0$ and $Y'' = 0$, so that from (3)

$$(5) \quad a + \beta y' + \gamma y'^2 + \delta y'^3 = 0.$$

This differential equation of the first order and third degree is satisfied by all the invariant straight lines. Assuming that there are four simple systems of such lines, then through each point there pass four; these give four values of y' , corresponding to fixed values of a, β, γ, δ . Equation (5) is therefore an identity. This means, however, that $Y'' = 0$ is a consequence of $y'' = 0$, or that $y'' = 0$ is invariant under T .* This proves the result stated above.

The result may be restated in terms of differential equations as follows. A simple infinity of straight lines is represented by a Clairaut differential equation

$$(6) \quad y = xy' + f(y'),$$

where y' is a one-valued function of x, y . If an equation of this kind is transformed into a similar equation, then the corresponding lines remain straight. Hence, if T transforms four Clairaut equations of the first degree into Clairaut equations, then the same is true of all Clairaut equations, and T is a collineation.

It may be added that what has been obtained is, in a sense, the minimum definitive property of the collineations. For there exist non-projective transformations which possess one, two, or three simple systems of invariant straight lines. A familiar example of the latter type is given by inversion with respect to a circle; here the three invariant systems are the pencils through the circular points at infinity and through the center of the circle. An example where all three systems are real is given by the quadratic transformation $X = 1/x, Y = 1/y$, which leaves invariant the systems $x = \text{const.}, y = \text{const.}, y/x = \text{const.}$ The same systems are invariant under the three parameter group $X = ax^c, Y = bx^c$.

The problem of determining all transformations possessing three invariant systems of straight lines is one of considerable difficulty. The writer expects to discuss it, as well as the extension of the results to space, in a future paper.

COLUMBIA UNIVERSITY.

* The direct integration of the equations $a = \beta = \gamma = \delta = 0$, leading to collineations, was accomplished by Scheffers; cf. p. 34, l. c. For infinitesimal transformations the analysis is simpler. Lie-Scheffers, *Differentialgleichungen*, 1891, p. 389.

A MODERN FRENCH CALCULUS.

Cours d'Analyse Mathématique. Par ÉDOUARD GOURSAT, Professeur à la Faculté des Sciences de Paris. Tome I. Gauthier-Villars, Paris, 1902.

THE revision of the fundamental principles of the calculus, which was initiated by Cauchy and Abel and carried through by Weierstrass and his followers, led to the development of the ϵ -proof (early introduced by Cauchy) and to the precise formulation of definitions and theorems. In Germany and Italy a tendency sprang up to place only such restrictions on definitions and theorems as are necessarily imposed by the nature of the case. Thus functions continuous throughout no interval whatever were admitted as the integrand of a definite integral simply because the form of the definition of the integral applied to a certain class of these functions, and the question was examined of how far the ordinary theorems of the integral calculus hold for such functions. Again, the theorem that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ was proved with fewer restrictions than the continuity of all the derivatives that enter. While this procedure is perfectly justifiable so far as it is a question of research in a special field, it is important not to lose sight of the fact that investigations of this sort are but a very special phase of modern analysis, and that even the specialist in the field of analysis may never need to trouble himself about the integrals of other functions than those which are continuous except at a finite number of points. That which is essential for every mathematician to know who has occasion to use the calculus to any extent is a simple formulation of the theorems and simple tests for the validity of the processes of the calculus which have been handed down to us from Euler's time and earlier : — when may a convergent series of continuous functions be integrated term by term, when may a definite integral whose integrand satisfies reasonable conditions of continuity be differentiated under the sign of integration? These are questions of general interest to mathematicians. To the importance of a simple and lucid answer French mathematicians are alive. With full appreciation of modern standards of rigor they do

not allow themselves to obscure in their presentation of the calculus the main facts of analysis by cumbersome details.

The book before us belongs to the best type of modern French treatises on the calculus. It is based on Professor Goursat's university lectures. According to the plan of instruction in France the student of mathematics learns at the *lycée* the meaning and use of derivatives, differentials not being introduced, and the rudiments of algebraic analysis.* Thus the university teacher can assume that the student is familiar with the notion of the limit and the elementary methods of the calculus, and that he has sufficient maturity to understand a treatment of the calculus such as is given in American universities in a second and third course.

After some introductory paragraphs, in which Rolle's theorem and the law of the mean are given, the author proceeds to a systematic treatment of partial differentiation, based on the total differential of a function. The fundamental theorems on which the properties of such differentials rest and which are overlooked in English text-books on the calculus† are here given their proper place.‡

Chapter II, pages 40–100, begins with the existence theorem for implicit functions, the proof being that which Dini gave in his lectures of 1877/78; and then follows the differentiation of the functions thus defined, properties of the Jacobian determinant, and change of variable. In algebra and algebraic analytic geometry the mere rudiments of partial differentiation suffice for the applications that arise; but in differential geometry and mathematical physics this is not the case. It is highly desirable that partial differentiation should be studied more at length than is at present the case, and a complete and lucid treatment such as is here given § will aid the teacher in modifying his course in that direction.

In American colleges students of calculus are not mature enough to appreciate existence theorems at the time when they

* Cf. Pierpont, "Mathematical instruction in France," BULLETIN (2), 6 (1900), p. 225.

† Cf. The writer's review: "A modern English calculus," BULLETIN (2), 8 (1902), p. 253.

‡ Cf. § 16, top of p. 29, and the corresponding theorem in § 14.

§ The present treatment is much fuller than that of Jordan, Cours d'analyse, and is illustrated by numerous applications of substantial character. A few examples taken from thermodynamics would have been a useful addition.

study partial differentiation and it is best to assume that equations defining implicit functions can be solved and lead to functions which have derivatives. It is easy, however, in using Professor Goursat's book, to assume these theorems at this stage and take them up later. In a logical development of the calculus they belong where the author has put them and for the working mathematician the arrangement here adopted is the most satisfactory one.

A few matters of detail before leaving these chapters. On page 5, line 12 from the top, the words "et à restes" should be inserted before "supérieure." The definition of approaching a limit in the case of a function of several variables is not given; the continuity of such a function is however carefully defined. The law of the mean for functions of several variables might well have been given a place in Chapter I. It appears nowhere explicitly, merely as a special case of Taylor's theorem with the remainder in Chapter III. On page 45, near the bottom, it is not of course true that if the three partial derivatives vanish simultaneously, the point is necessarily a singular point. This oversight occurs at various other places in the book. In the theorem at the beginning of § 28 it is necessary for the proof that for the system of values of $(u_1, \dots, u_n)$ considered the partial derivatives $\Pi_{u_1}, \dots, \Pi_{u_n}$ should not vanish simultaneously. This requirement should be made in the statement of the theorem. In this section (§ 28) the author does not bring out with all the clearness one could wish the fact that there are in all three cases: (a) the case in which the Jacobian determinant does not vanish; (b) the case in which it vanishes identically; and (c) the case in which it vanishes at a given point, but not at all points in the neighborhood. Cases (a) and (b) are treated. But little is known about case (c), and there is no reason for considering it; but the classification is important. A large number of examples are given at the end of Chapters I and II, all of which are taken from life. While the student will need to do many easy examples, like those given in English books on calculus, at the start, he will have the satisfaction in working these more difficult ones of knowing that they are not artificial, that they truly illustrate the actual applications of partial differentiation.

Chapter III is devoted to Taylor's theorem with the remainder and to Taylor's series, both for functions of one and for functions of several variables, and to applications. The

French have not committed themselves to the exclusive use of power series.* When it is simpler to use Taylor's theorem with the remainder, they do so; and it is an excellent feature of the book before us that the infinite series is not employed when the finite series is better suited to the purpose. Indeterminate forms are treated by the use of Taylor's theorem, but we miss the more general theorem that if $f(a) = \phi(a) = 0$ or ∞ , and if $f'(x)/\phi'(x)$ approaches a limit, then $f(x)/\phi(x)$ approaches a limit and $\lim f(x)/\phi(x) = \lim f'(x)/\phi'(x)$. Thus such limits as

$$\lim_{x=0} x \log x \quad \text{or} \quad \lim_{x=\infty} \frac{\log x}{x},$$

or more generally

$$\lim x^\alpha (\log x)^\beta,$$

when $x = 0$ or ∞ ; or again

$$\lim_{x=\infty} x^n e^{-x}$$

must each be evaluated by a special investigation,—one for which no general method is given.

The treatment of maxima and minima of functions of several variables is admirable, adequate and clear, but not overdone.† The examples studied are well chosen and include clothed problems that do not come under the ordinary rule.

Chapters IV–VII, pages 150–368, are entitled respectively Definite Integrals, Indefinite Integrals, Double Integrals, and Multiple Integrals, Integration of Total Differentials. In American colleges the custom is growing of introducing the definite integral, defined as the limit of a sum and represented by the area under a curve, early in the first course in calculus, and of applying it to the solution of a variety of problems in physics and mechanics. Thus the student comes to the second course in calculus with a pretty good idea of what integrals are

* This tendency in Germany is well hit off by Bohlmann in his report on important works on the calculus, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 6 (1897), p. 110. One of the entries in the table of contents reads "Es gibt nur Potenzreihen!"

† The definition of a maximum or a minimum on page 130 would seem to exclude the case that $f(c+h) - f(c)$ can vanish for other values of h than the value $h=0$; but this is not the definition employed on page 136: "Le cas où $b^2 - ac = 0 \dots$ "

and what they are for, and he is prepared for the study of double integrals, improper integrals and functions defined by integrals. It is at this stage, that is, in the second calculus course in American colleges, that these chapters of Professor Goursat's book may be employed with great advantage. They give a simple and rigorous elementary treatment of the subjects just mentioned — particularly the chapter on double integrals, pages 282–333 — a masterpiece of presentation, so clear and rigorous and to the point that it may well serve as a text in the treatment of this subject. Regarding the first of these chapters a similar remark applies to the one made concerning the existence theorem for implicit functions in Chapter II; namely, that in the second course in calculus it is well to assume the theorems about continuity, the proof coming more properly at a later stage, when questions of uniform convergence are treated; but here again the arrangement which the author has adopted will commend itself to the working mathematician as being the natural sequence of topics. There is one omission that seems to us unfortunate. It is that of Duhamel's theorem regarding the representation of any infinitesimal in a limit of a sum by another infinitesimal that differs from it by one of higher order. But as we are treating this subject at length elsewhere, we restrict ourselves here to a reference.*

The following topics considered in these chapters deserve special mention because of the lucid and rigorous treatment: change of variables in simple and multiple integrals §§ 84, 128,† 130, 145, 150; improper integrals simple and multiple,

* *Annals of Math.* (2), 4 (July, 1903).

† The method here set forth of proving by means of Green's theorem that when curvilinear coördinates $x=f(u, v)$, $y=\phi(u, v)$ are introduced, the area Ω bounded by the curves $u=u_0$, $u=u_0+du$, $v=v_0$, $v=v_0+dv$ is

$$\Omega = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| du dv,$$

where the Jacobian is to be formed for a certain point (u', v') in Ω , is especially elegant. The author's reason for not extending it to triple integrals was doubtless that the formulas become a little long; but they present no serious difficulty and the generalization may well serve as an exercise for the student, his attention being first called to the fact that the three-rowed Jacobian may be written in the form

§ 89*-91, 133; line and surface integrals, including Green's and Stokes's theorems, §§ 93, 126, 135, 136, 149, 151-155: differentiation under the sign of integration, § 97; approximate calculation of an integral by Simpson's and Gauss's rules; Amsler's planimeter, § 99-102; the geometric interpretation of the method of rationalization employed for computing the integrals $\int R(x, \sqrt{a + bx + cx^2}) dx$, § 105.

Chapters VIII, IX, pages 369-478, deal with infinite series. The subject is treated *ab initio* and the ordinary tests for convergence and theorems relating to the algebraic transformation of series are developed with the clearness and rigor that mark the whole book. But the last page of § 169 relating to double series whose terms are not all of like sign needs to be expanded. — The definition of uniform convergence in the case of infinite series is the one given by Darboux,† and is as follows: The series of continuous functions

$$u_0(x) + u_1(x) + \dots,$$

convergent in the interval (a, b) , is said to be uniformly convergent in this interval if to every positive ϵ there corresponds a positive integer n , independent of x , such that the remainder

$$R_n(x) = u_n(x) + u_{n+1}(x) + \dots,$$

remains in absolute value less than ϵ for all values of x in the

$$\begin{vmatrix} f_u & f_v & f_w \\ \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \end{vmatrix} = \frac{\partial}{\partial u} \begin{vmatrix} f_v & f_w \\ \phi_v & \phi_w \end{vmatrix} + \frac{\partial}{\partial v} \begin{vmatrix} f_u & f_w \\ \phi_u & \phi_w \end{vmatrix} + \frac{\partial}{\partial w} \begin{vmatrix} f_u & f_v \\ \phi_u & \phi_v \end{vmatrix}.$$

The possibility of this extension was pointed out by the author in his original paper, *Bulletin des sciences math.* (2), 18 (1894), p. 72.

* The theoretical developments here given are especially simple. The rule for the convergence of the integral

$$\int_a^b f(x) dx,$$

when $f(x)$ is discontinuous at the point $x = a$, page 198, bottom, can, however, be formulated still more simply if we require merely that for some value of $0 < k < 1$ the variable $(x-a)^k f(x)$ shall approach a limit when x approaches a . A similar remark applies to the formulation of the other tests for convergence and divergence of improper integrals.

† *Ann. École norm.* (2), 4 (1875), page 77.

interval. It will be observed that this definition differs from the one ordinarily given in that it does not require that $|R_n(x)| < \epsilon$ for every $n > m$, but only for one value of n . Nevertheless the proof that such a series represents a continuous function is sound. The proof that the series may be integrated term by term is, however, invalid, as the following example shows: let

$$S_n(x) = nxe^{-nx^2} \text{ when } n \text{ is odd};$$

$$S_n(x) = 0 \quad \text{“} \quad n \text{ “ even.}$$

Here the term by term integral in the interval $(0, 1)$ has the value

$$\begin{aligned} \int_0^1 S_n(x) dx &= \frac{1}{2}(1 - e^{-n}) \text{ when } n \text{ is odd}; \\ &= 0 \quad \text{“} \quad n \text{ “ even.} \end{aligned}$$

What is true is this: if such a series be integrated term by term and parentheses then be suitably introduced, the series of parentheses will converge toward the integral of the function as its limit. Tannery * has called attention to the fact here mentioned, namely, that a series uniformly convergent according to Darboux's definition can not always be integrated term by term, but both Picard † and Goursat appear to have overlooked Tannery's corrections. In § 175, however, which deals with the differentiation of an improper integral

$$\int_a^\infty f(x, \alpha) dx$$

under the sign of integration, the definition of the uniform convergence of such an integral is the one ordinarily given. If such an integral converges uniformly (and the integrand is continuous in x, α), it represents a continuous function of α and may be integrated under the sign of integration:

$$\int_a^{a_1} d\alpha \int_a^\infty f(x, \alpha) dx = \int_a^\infty dx \int_{a_0}^{a_1} f(x, \alpha) d\alpha.$$

These theorems are precisely analogous to the theorems about

* Fonctions d'une variable, Paris, 1886, p. 366, foot-note. The foot-note of pages 133-4, so far as it relates to integration, contains an inaccuracy, which is corrected in the later foot-note just cited.

† Traité d'analyse, vol. I, chap. VIII, 1st ed., p. 195; 2d ed., p. 211.

uniformly convergent series and should be given along with the sufficient condition the author deduces for differentiating under the sign of integration. The theorems relating to the transformation of power series in one and in several variables, and the theorems that follow from these, together with the proof of the analytic character of the implicit functions defined by a system of analytic equations, $F_i(x_1, \dots, x_p; y_1, \dots, y_p) = 0$ ($i = 1, \dots, p$) are all faithfully developed, pages 419–461 being devoted to this topic. The method of the *fonctions majorantes* is explicitly set forth. Analytic functions of real arguments are defined by the property that they admit a development by Taylor's theorem. The author does not neglect to remind the reader that in spite of the important rôle that these functions play, they form after all but a very special group of real functions of real variables within the general group of continuous functions. Finally the development of a continuous function into a Fourier's series is established. The volume ends with three chapters, pages 479–610, on applications of the differential calculus to curves and surfaces.

The extraordinarily high standard of simplicity and attractiveness in style, combined with modern rigor, which Picard set in his *Traité d'analyse* is fully maintained by Professor Goursat. The objects of the two works are quite different. Picard's purpose was to write a treatise on differential equations, and he developed only such parts of analysis and geometry as bear on this subject. Goursat, on the other hand, has set himself the task of writing a systematic treatment of the calculus, and thus the whole field of the calculus is included here.

A treatise on advanced calculus which should present the whole subject rigorously and attractively, and at the same time in the spirit of modern analysis, has been sorely needed by students of mathematics who intend to proceed to the study of mathematical physics or of some of the various branches of analysis — theory of functions, differential equations, calculus of variations, etc. Professor Goursat's work meets the needs of such students in a thoroughly satisfactory manner, and we recommend it to them most heartily. The teacher of calculus will find many suggestions in the book which will enable him to improve his course, and he may often with advantage refer even an elementary class to the more elementary parts of the book for collateral reading. The range of the book is wide.

While beginning with the elements of the calculus, it carries the reader to the point where he is prepared to use original sources and extracts from ϵ -proofs the underlying thought. When the future historian inquires how the calculus appeared to the mathematicians of the close of the nineteenth century, he may safely take Professor Goursat's book as an exponent of that which is central in the calculus conceptions and methods of this age.

W. F. OSGOOD.

HARVARD UNIVERSITY,
February, 1903.

SHORTER NOTICES.

Niedere Zahlentheorie. Erster Teil. By Dr. PAUL BACHMANN. B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band X, 1. Leipzig, 1902. x + 402 pp.

IN view of the ambitious series of volumes by Bachmann, giving a comprehensive exposition of number theory, a series not yet completed, the appearance of a new volume on the elements of the subject, quite independent of the series mentioned, will doubtless cause some surprise. When the invitation came to the author to contribute to Teubner's Sammlung a text upon the subject on which he is so eminent an authority, he hesitated long, fearing that a text on the elements of number theory ran the risk of conflicting with his Elementen. The author has attempted to avoid this conflict in two directions: first by the addition of much important material; second, by employing a method of construction different at least in essential points. The author believes that the present book, both in contents and in foundation, may well be considered as a supplementary volume to his former series. As indicating in detail parts differing essentially from the Elementen, there may be mentioned the chapter on the different euclidean algorithms, including Farey's series, the theory of binomial and general congruences, the exhaustive treatment of the known proofs by elementary number theory of the quadratic reciprocity law and the interrelations of these proofs. The theory of higher congruences is appropriately introduced, even in the *Niedere Zahlentheorie*, both by way of climax to the elementary parts and to afford a satisfactory insight into the means employed by Gauss in his seventh proof of the reciprocity law.

The introduction (15 pages) gives a brief history of number theory. Chapter I is a rather formal discussion of the concept number, the idea of sequence being taken for the foundation. The 38 pages of Chapter II are devoted to the divisibility of numbers. On pages 34 and 35 is given a simple proof of the existence of integers ξ and η satisfying the equation $a\xi + b\eta = \delta$, where δ is the greatest common divisor of a and b , use being made of the *modulus* of Dedekind and Kronecker. But this proof of the fundamental theorem gives no algorithm for finding ξ and η as does the proof based on Euclid's algorithm. A series of special recent theorems due to Weill, André, de Polignac, Catalan, Landau, and Liouville show that certain expressions in fractional form have integral values. The 33 pages of Chapter III are devoted to the usual theorems on residues and congruences. The 54 pages of Chapter IV are devoted to the euclidean algorithm, continued fractions, Farey's series. The 27 pages of Chapter V relate to the theorems of Fermat and Wilson and their generalization. Chapter VI, on the theory of quadratic residues, is the longest in the book, containing 138 pages; it is devoted chiefly to the various proofs of the reciprocity law, a chronological table of which appears on pages 203 and 204, with an addition on page 402. The 82 pages of the final Chapter VII gives the theory of higher congruences. The theory of primitive roots is treated at length with two distinct proofs of the existence of primitive roots modulo p , a prime. A general theory, including that of circulating decimals, is then developed. Finally follows the chief results of Galois, Schönemann, Dedekind, and Serret on higher congruences. On pages 373-375 is given Gauss's elegant and simple method of determining the number of incongruent primary irreducible functions (mod p) of degree n ; following this is the usual purely arithmetic development.

The author has certainly succeeded admirably in the task he has undertaken. It will appeal especially to the student who wishes a complete account of the results and methods of the elementary parts of the theory of numbers.

L. E. DICKSON.

The Constructive Development of Group Theory, with a Bibliography. By BURTON SCOTT EASTON. Philadelphia, Publications of the University of Pennsylvania. Mathematics, No. 2, 1902. 8vo, iv + 89 pp. Ginn & Co., Selling Agents.

"THE purpose of this monograph is to present in a consecutive form the principal features of abstract and substitution group theory." It has been the aim of the author "to examine in detail all memoirs dealing directly with such group theory (excluding, in particular, that of linear groups) and to construct from this material a continuous treatise on the subject." These few words from the preface are sufficient evidence of the importance and difficulty of the task which the author has set for himself.

The monograph consists of two parts — a bibliography, covering pages 5–38; and a compendium, covering pages 39–86. The former is preceded by a prefatory note and a list of periodicals with abbreviations, while the latter is followed by a good subject index. The bibliography includes 157 names and gives under each name the titles and places of publication, arranged in chronological order. As this is the first publication of its kind, it is a very welcome addition to the literature on the subject and will doubtless prove valuable to all workers in this field.

Although great care seems to have been exercised in compiling the bibliography, yet it can scarcely be expected that the first edition of such a comprehensive collection would be free from errors. A few corrections are made in the addenda which closes the memoir. To these should be added Tanner's paper "On the enumeration of groups of totitives," *Proceedings of the London Mathematical Society*, volume 27; Hurwitz, "Algebraische Gebilde mit eindeutigen Transformationen in sich," *Mathematische Annalen*, volume 41; Study, "Sphärische Trigonometrie," *Leipziger Abhandlungen*, volume 20; the algebras of Comberousse and Pincherle, and possibly one or two of the papers of H. Laurent.

Many readers will doubtless regret that the author did not aim to include all papers on linear groups. This subject is so closely related to that of abstract and substitution groups that it frequently seems difficult to decide whether a paper on one of these subjects has sufficient bearing on the other to be classed with its literature. Such an extension would however have made the task of the author still more difficult, and he

can scarcely be blamed for leaving this part to some one else. We may note here a slight typographical error. Under M 24 the word "where" should be replaced by *whose*.

The second part practically consists of the statement of theorems without proof and a comprehensive list of references to works where proofs may be found. These references are arranged in chronological order and tend to make this part even more useful than the first. By employing abbreviations very freely a vast amount of information is crowded on less than fifty pages. It is to be hoped that many of the readers will have the kindness to comply with the author's request and notify him of any inaccuracies which they may observe. The following have been observed by the reviewer :

In no. 134 it should be stated that g is a power of a prime. Otherwise the theorem is not generally true. With this change the theorem belongs under heading 9, "Groups whose order is a power of a prime number." As a special case of this theorem no. 119 should read $m > 6$ instead of " $m > p + 4$." Of course, the latter is true, but it is not as general as the former. In no. 142 the term "subgroup" at the end of the first line should be replaced by *group*, and in no. 146 the expression "two abelian subgroups" should be replaced by *an abelian group*. In no. 166 for $n - 1$ read $m - 1$, and replace "primitive group" by *primitive perfect group* in 168.

In nos. 171-173 it should be stated that the primitive groups in question do not include the alternating group and *greater than 3* should be added to the first line of no. 175. A curious oversight is presented by no. 219. The holomorph of G is never the direct product of G and its i -group. In no. 20 the exponents should evidently be $m_1, m_2, \dots$ in place of $m_1, m_2, \dots$

According to Cantor's dictum * "*da die Geschichte unwiderstehlich die Veröffentlichungszeit als allein massgebend betrachten muss, wo Erstlingsrechte zu vergeben sind*" it would appear that the expression "Due to Dd" under no. 103 should either be omitted or replaced by *Due to M.*† The references under no. 105 are somewhat misleading as the theorem was first given in this form by M 13,267. Its proof was directly based upon older theorems and due acknowledgment was made in the first publication.

G. A. MILLER.

* Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. 2 (1900), p. 811.

† Frobenius, *Berliner Sitzungsberichte* (1896), p. 1348.

Énumération des Groupes d'Opérations d'Ordre Donné. Par RAYMOND LE VAVASSEUR, à Toulouse. Paris, Hermann. 4to, pp. 128, lithographed. No index or title-page. Fr. 5.00.

THE appearance of a new treatise on groups has ceased to be an event in the mathematical world, but M. Le Vavasseur's work is something a little out of the ordinary. For by means of nine chapters devoted respectively to groups of orders p , p^2 , pq , 8 , pq^2 ($p > q$), 16 , p^2q , $8p$, p^3 and pqr its purpose is to introduce persons unfamiliar with group theory directly to the problems of group construction. And hence everything in each of these chapters is directed towards answering the question, "How many groups are there of the assigned order and what are their equations?"

No knowledge whatever of the subject is assumed. The author begins with the fundamental definitions of the theory and develops the necessary preliminary propositions in a manner that follows Burnside's treatment with considerable exactness, introducing, for instance, into the French terminology such phrases as *suite complète*, *opération conjuguée d'elle même*. The whole of the second chapter is adapted from Burnside—so completely that the writer might have expressed an acknowledgment of his source more fully than in the single reference on page 16. These chapters and those immediately following are devoted to the simpler cases of group construction, too familiar ground to admit of much originality in treatment, but the discussion given is lucid and tolerably concise. It is certainly less prolix than Hölder's standard paper in volume 43 of the *Mathematische Annalen*. But this virtue of brevity cannot be claimed for the analysis of groups of order $8p$, as this chapter fills sixty-four pages and embodies a treatise on the group of automorphisms of abelian groups of order p^3 and type $(1, 1, 1)$, with the representation of such groups in linear form and an excursus on bilinear substitutions. While the developments are interesting and while the results are perfectly correct (albeit the conclusions for order 56 are not very clear), yet the results could be attained in a very small fraction of the space used—cf., e. g., Professor Miller's paper in volume 43 of the *Philosophical Magazine*.

All orders less than 30 are discussed with complete thoroughness, and the groups for each case are determined and their equations given. A comparison with Hölder's results seems to

show a simpler form of the defining relations, but inasmuch as the simplification is gained by using generators of a higher order its advantage is more or less a matter of taste. The arrangement of the tables is good, but the number of groups for a particular order is not always quite obvious.

The chief defect of the book, however, is its plan. It is evidently intended for beginners, but to set beginners immediately at work on group types is a procedure of very dubious wisdom. The methods involved are necessarily mechanical and monotonous—monotonous in that they demand the solution of a large number of congruences that differ but slightly, and mechanical in that the solution of such congruences adds little to the mathematical knowledge of the student. Moreover, the particular methods employed do not seem well adapted for extension to groups of a higher order—if the chapter on order $8p$ were a fair criterion, more complex groups would necessitate a prohibitive amount of labor. The last chapter however—on orders free from a quadratic factor—adopts Frobenius's method of a congruence with a double modulus, but terminates so abruptly that the author evidently intends to complete it later.

The book is unusually easy to read. Every needed theorem from other branches of mathematics is always fully stated and generally proved. And to those who are studying relations between abstract and linear groups the book may be recommended without hesitation.

B. S. EASTON.

Elemente der Theorie der Determinanten mit vielen Uebungsaufgaben. Von Dr. P. MANSION. Dritte vermehrte Auflage. Leipzig, Teubner, 1899. 103 pp.

THIS book, which has passed through six French editions, is an excellent introduction to the more extended treatises of Baltzer, Pascal, and Gordan. Its scope is distinctly elementary. The more complicated developments of determinants, as Laplace's, are passed over with a statement of fact without proof while functional determinants, and the general development with respect to a row and column simultaneously are not mentioned.

The book opens with an introduction of twenty-two pages on determinants of the second and third orders, which are illustrated with a large number of problems. The usual discussion of the elementary properties of the general determinant follows. The points of greatest pedagogical interest are, first, the admir-

ably clear and thorough account of the special cases that occur in the solution of linear systems and their dependence, second, the treatment of Sylvester's dialytic method of elimination in which not only the necessity but the sufficiency of the condition that two equations have λ common roots is derived — a point in which most elementary texts are painfully deficient. A brief and hence necessarily unsatisfactory note appears in the appendix on the development of the properties of determinants by means of alternate numbers, a method which is extremely rapid, and with somewhat mature students quite teachable.

No geometrical applications are given, except in isolated examples. Problems are abundant, well graded and not trivial. Not only is the book sufficiently elementary to warrant its wide use as a text-book, but its many pedagogical excellencies make it very valuable to a teacher of the subject.

H. E. HAWKES.

Liniengeometrie mit Anwendungen. Band I. By Dr. KONRAD ZINDLER, Professor an der Universität Innsbruck. Sammlung Schubert, XXXIV. Leipzig, G. J. Göschen, 1902. 8vo., viii + 380 pp., 87 figures.

IN this systematic exposition of line geometry, both synthetic and analytic methods are employed. It is the first book of this character since the appearance of Plücker's original treatise.* In fact, Sturm's *Liniengeometrie* in three volumes (1892, 1893, 1896) is purely synthetic; while other books, such as Koenigs's *Géométrie réglée*, treat only special parts of the subject.

Line geometry treats of manifoldnesses of straight lines in space. The names ruled surface, congruence, complex are applied to a one-fold, two-fold, three-fold system, respectively.

The present (first) volume treats chiefly of linear complexes and congruences and their applications; the second volume will be concerned chiefly with algebraic line configurations of higher than the first degree. The book is intended as a text-book; eighty-two exercises are scattered through the book, hints for the solution of certain of which are given in the appendix, with references for the others.

Chapter I treats of the Nullsystem and the Strahlengewinde. Let all the points of space be subjected to a given screw motion

* *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement*, I, 1868; II, 1869.

and let each point be made to correspond to that plane through it which is normal to the path of the motion. Inversely, to every plane corresponds a point. This one-to-one geometric correspondence is called a Nullsystem. If a point describes a straight line, the corresponding plane turns about another line, and inversely. Moreover, the correspondence preserves incidences. Hence a Nullsystem is a correlation in the sense of projective geometry. The totality of normals, at all points of space, to the path of a screw motion forms a Strahlengewinde; it is composed of ∞^3 straight lines; every point of space and every plane of space is the bearer of a flat-pencil of these lines. A simple visualization of the lines of a Strahlengewinde may be obtained by taking certain ∞^2 tangents to each of ∞^1 coaxial right circular cylinders.

Chapter II treats at length (pages 31–71) of applications to the theory of motion, mechanics and graphic statics. Chapter III treats in fifty-five pages the subject of line coördinates and the coördinates of a segment (Stab) defined by the length and the position of the line on which the sect lies. By way of introduction, the chief properties of homogeneous point and plane coördinates are given without proof.

Chapter IV treats of linear configurations of lines with applications to mechanics. A linear homogeneous equation between the six segment coördinates defines a linear complex. It is shown to be identical either with a Strahlengewinde or else with the totality of lines intersecting a given one. Hence the linear complex can be represented by a linear homogeneous equation in line coördinates. Its coefficients define a correlation in space for which every point lies in its corresponding plane; inversely, every such correlation may be defined by a Strahlengewinde. The section of two linear complexes is a congruence of the first order and first class, called (after Sturm) a net of lines. The properties of nets, including a complete classification, are given on pages 160–197.

Chapter V (pages 203–278) treats of imaginary elements, and their geometric interpretation, including a historical and logical discussion. The final chapter, VI, relates to manifoldnesses of linear complexes and applications to mechanics and the theory of motion. Among the topics treated may be mentioned the cylindroid (a ruled surface of the third order), general complex coördinates, and Klein line coördinates.

L. E. DICKSON.

NOTES.

THE July number (volume 4, number 3) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On the point-line as element of space: a study of the corresponding bilinear connex," by E. KASNER; "On the formation of the derivatives of the lunar coördinates with respect to the elements," by E. W. BROWN; "On reducible groups," by S. EPSTEIN; "Theory of linear associative algebra," by J. B. SHAW; "Projective coördinates," by F. MORLEY; "On an extension of the 1894 memoir of Stieltjes," by E. B. VAN VLECK; "On the variation of the arbitrary and given constants in dynamical equations," by E. W. BROWN; "The primitive groups of class $2p$ which contain a substitution of order p and degree $2p$," by W. A. MANNING; "Complete sets of postulates for the theory of real quantities," by E. V. HUNTINGTON.

THE July number (volume 25, number 3) of the *American Journal of Mathematics* contains: "Isothermal-conjugate systems of lines on surfaces," by L. P. EISENHART; "Some differential equations connected with hypersurfaces," by G. O. JAMES; "On the forms of sextic scrolls of genus greater than one," by VIRGIL SNYDER; "Geometry on the cuspidal cubic cone," by F. C. FERRY.

THE concluding (July) number of volume 4 of the *Annals of Mathematics* contains: "On the characteristics of differential equations, Part II," by E. R. HEDRICK; "On the uniformity of the convergence of certain absolutely convergent series," by M. BÔCHER; "The integral as the limit of a sum, and a theorem of Duhamel," by W. F. OSGOOD; "On a general relation of continued fractions," by R. E. MORITZ; "Note on the equilateral hyperbola," by J. A. VAN GROOS; "A new proof of the generalized Wilson's theorem," by G. A. MILLER; "A sufficient condition for the maximum number of imaginary roots of an equation of the n th degree," by E. B. VAN VLECK.

AT the Fourth Colloquium, to be held in connection with the coming summer meeting of the AMERICAN MATHEMATICAL

SOCIETY (cf. previous announcement, BULLETIN, current volume, page 384) Professor H. S. WHITE will give a course of three lectures on "Linear systems of curves on algebraic surfaces." The Colloquium will open on Wednesday morning, September 2, and close on the following Saturday morning. The twelve lectures given in the three courses will be distributed as follows: two on each morning, Wednesday to Saturday inclusive, and two on each of two afternoons. The usual charge of three dollars will be made to those attending any or all the courses.

At the meeting of the London mathematical society held on April 16, 1903, the following papers were read: By Mr. C. S. JACKSON: "Exhibition of the logo-logarithmic slide rule"; by Mr. R. F. GWYTHYER: "On the deduction of Schlömilch's series from a Fourier series, and its development into a definite integral"; by Mr. E. T. WHITTAKER: "On those functions which are defined by definite integrals with not more than two singularities"; by Professor A. E. H. LOVE: "Note on exact solutions of the problem of the bending of an elastic plate under pressure"; by Professor A. LODGE: "Relations between points (in a plane) having conjugate complex coördinates."

At the meeting held on May 14, the following papers were read: By Dr. H. F. BAKER: (1) On the definiteness of quadratic forms with imaginary coefficients; (2) On a certain form of logical argument which occurs in the proofs of several fundamental theorems of pure mathematics; (3) On the summation of Neumann's series representing a potential defined by boundary values; (4) On the formation of the variant equation in the theory of differential equations; (5) On some points in the theory of continuous groups; by Lieut.-Col. A. CUNNINGHAM: Announcement of the discovery made jointly by Rev. J. Cullen, Mr. A. E. Western, and the author, of seven new factors of Fermat's numbers ($2^{2^n} + 1$); by Mrs. YOUNG: The surface representing all right-angled spherical triangles; by Mr. W. H. BUSSEY: Generational relations defining an abstract simple group of order 32,736; by Mr. W. H. YOUNG: (1) Skew surfaces contained in a linear congruence; (2) Closed sets of points and Cantor's numbers.

THE National academy of sciences held its annual meeting at Washington, D. C., April 21-23. No mathematical papers were presented at this meeting. Officers for the ensuing year are

as follows: president, ALEXANDER AGASSIZ; vice-president, IRA REMSEN; foreign secretary, SIMON NEWCOMB; home secretary, ARNOLD HAGUE; treasurer, S. F. EMMONS. Professor NEWCOMB was also appointed delegate to represent the Academy at the meeting of the international association of academies held at London in June. Professor E. PICARD, of the University of Paris, was elected foreign associate of the Academy.

At its second general meeting, held at the Armour Institute of Technology, Chicago, April 10–11, the Central association of physics teachers was reorganized as the Central association of science and mathematics teachers. The mathematical section held two sessions, devoted to the discussion of improved methods of teaching elementary mathematics. One of the sessions was held jointly with the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY, as noted in the Secretary's report, page 533 of the present number of the BULLETIN.

THE German mathematical society is making active preparations for the third international congress of mathematicians to be held at Heidelberg, August 8–13, 1904, under the presidency of Professor H. WEBER. The congress will divide into six sections devoted respectively to arithmetic and algebra, analysis, geometry, applied mathematics, history of mathematics, and teaching of mathematics. Exhibitions of models and of mathematical literature have been added to the programme. The government of Baden has appropriated 3,000 marks toward the financial support of the congress. Cards entitling the holder to general participation in the various events of the congress will be issued on payment of 20 marks. Each holder is then entitled to obtain additional cards for members of his family for 10 marks each. A daily bulletin will be issued during the meeting, giving besides the day's programme a list of those in attendance and their lodgings.

THE universities below offer during the academic year 1903–1904 advanced courses in mathematics as follows:

COLUMBIA UNIVERSITY. — By Professor T. S. FISKE: Advanced calculus, three hours; Theory of functions of a complex variable, three hours. — By Professor F. N. COLE: Theory of groups, three hours; Theory of invariants, three hours. — By Professor R. S. WOODWARD: Advanced theoretical mechanics, two hours; Theory of the potential function, two hours; Mathe-

mathematical theory of elasticity, two hours. — By Professor D. E. SMITH: History of mathematics, two hours. — By Professor J. MACLAY: Application of the calculus to surfaces and curves in space, three hours. — By Professor C. J. KEYSER: Modern theories in geometry, three hours. — By Dr. G. H. LING: Infinite series and products, two hours. — By Dr. E. KASNER: Transformations and continuous groups, two hours.

JOHNS HOPKINS UNIVERSITY. — By Professor F. MORLEY: Higher geometry, three hours; Vector analysis, two hours, first half year; Kinematics, two hours, second half-year. — By Dr. A. COHEN: Partial differential equations, two hours; Differential geometry, two hours, first half-year; Theory of numbers, two hours, second half-year; Elementary theory of functions, two hours.

THE memorial of the late Professor Sir G. G. STOKES, planned by the Royal Society and Cambridge University, is to take the form of a medallion portrait to be placed in Westminster Abbey.

THE Adams prize for 1903 (see BULLETIN, volume 7, page 321) was not awarded.

PROFESSOR M. NOETHER, of the University of Erlangen, has been elected corresponding member of the Paris academy of sciences.

PROFESSOR G. MITTAG-LEFFLER has been elected honorary member of the Finnish scientific society of Helsingfors.

THE Redpath professorship of mathematics at McGill University has been filled by the appointment of Professor JAMES HARKNESS, of Bryn Mawr College. Dr. H. M. TORY, of McGill University, has been promoted to an associate professorship of mathematics.

PROFESSOR FLORIAN CAJORI, of the department of mathematics at Colorado College, has been elected dean of the school of engineering of that institution. He has also been appointed one of the American representatives of the international commission on the organization of a congress of the history of sciences, referred to on page 509 of the current volume of the BULLETIN.

AT Purdue University Professor A. M. KENYON has been promoted to a full professorship of mathematics. Mr. W. H. BATES has been appointed instructor in mathematics.

At the University of Nebraska, Dr. C. C. ENGBERG has been promoted to an adjunct professorship of mathematics.

At Cornell University Dr. J. I. HUTCHINSON and Dr. VIRGIL SNYDER have been promoted to assistant professorships of mathematics.

MR. C. H. ASHTON has been appointed to an assistant professorship of mathematics in the University of Kansas.

MR. L. D. AMES has been appointed to an instructorship in mathematics in the University of Missouri.

MR. CLIFFORD GRAY and Mr. E. A. HOOK have been appointed instructors in mathematics at Columbia University.

DR. CARL STROMQUIST has been appointed to an instructorship in mathematics in the Greene School of Science of Princeton University.

MR. A. H. WILSON, of Princeton University, has received leave of absence and will spend the coming year at the University of Greifswald.

THE death of Professor C. A. BJERKNES, of the University of Christiania, is announced.

PROFESSOR L. GEGENBAUER, of the University of Vienna, died on June 3, at the age of fifty-four.

THE following catalogues of second-hand mathematical works have recently appeared : Burgersdijk and Niermans, Leyden, a few entries ; Adolf Geering, Bäumleingasse 10, Basel, catalogue no. 175, a few entries ; Ferdinand Schöningh, Lortzingstr. 2, Osnabrück, catalogue no. 43, 176 entries ; W. Junk, Berlin, N. W., 5, a catalogue of 42 pages, representing a library of mathematical, physical, and astronomical works to be sold *en bloc* for 10,000 marks ; Oskar Schack, Königsstrasse 15, Leipzig, catalogue no. 98, 401 entries in mathematics and physics ; Paul Geuthner, 10 rue de Buci, Paris, catalogue no. 5, 241 entries, writers before 1800, also 91 portraits ; Oskar Gerschel, 16 Calwerstr., Stuttgart, catalogue no. 27, 480 entries.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADAM (C.). See DESCARTES.

BIANCHI (L.). Lezioni sulla teoria dei gruppi continui di trasformazioni. Pisa, 1903. 8vo. 708 pp. (Lith.). Fr. 16.00

BOUCHER (M.). Essai sur l'hyperespace, le temps, la matière et l'énergie. Paris, Alcan, 1903. 16mo. 208 pp. Fr. 2.50

BÜRKLEN (O. T.). Formelsammlung und Repetitorium der Mathematik, enthaltend die wichtigsten Formeln und Lehrsätze der Arithmetik, Algebra, algebraischen Analysis, ebenen Geometrie, Stereometrie, ebenen und sphärischen Trigonometrie, mathematischen Geographie, analytischen Geometrie der Ebene und des Raumes, der Differential- und Integralrechnung. 2te Auflage, 4ter Abdruck. Leipzig, Göschen, 1903. 12mo. 229 pp. Cloth. (Sammlung Göschen, No. 51.) M. 0.80

CRESCOEUR (A. J. M.). Cours d'analyse. Calcul différentiel et calcul intégral. Méthode simple. Anvers, 1902. 8vo. 336 pp. Fr. 5.50

DELAPORTE (L. J.). Essai philosophique sur les géométries non-euclidiennes. (Thèse.) Paris, Naud, 1903. 8vo. 119 pp.

DESCARTES. Oeuvres de Descartes. Publiées par Charles Adam et Paul Tannery, sous les auspices du ministère de l'instruction publique. Vol. 5: Correspondance, mai 1647-février 1650. 669 pp. Vol. 6: Discours de la méthode et Essais. 12 + 730 pp. Paris, Cerf, 1902-1903. 4to.

DU BOIS (H. E. J. G.). Het verband tuschen wiskundige, proefondervindelijke en toegepaste natuurkunde. Utrecht, 1902. 8vo. 44 pp. M. 1.20

FORSYTH (A. R.). Treatise on differential equations. 3d edition. London, Macmillan, 1903. 8vo. 528 pp. Cloth. 14s.

FRIZZO (G.). De numeris libri duo authore Ioanne Noviomago, esposti ed illustrati. Appendice. Verona e Padova, Drucker, 1903. 8vo. 25 pp.

HOLST (E.). Om høiere arithmetiske raekker samt nogle af de almindeligst forekommende konvergerende raekker, med inledende saetninger om den hele funktion. 4te oplag. Christiania, 1902. 8vo. 55 pp. Boards. M. 1.50

KOENIGSBERGER (L.). Hermann von Helmholtz. Vol. II. Braunschweig, Vieweg, 1903. 8vo. 16 + 383 pp., 2 portraits. Cloth. M. 10.00

——. Vol. III. Braunschweig, Vieweg, 1903. 8vo. 10 + 142 pp., 4 portraits, 1 facsimile. Cloth. M. 9.00

KRAZER (A.). Lehrbuch der Thetafunktionen. Leipzig, Teubner, 1903. 8vo. 24 + 512 pp. Cloth. (B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften, mit Einschluss ihrer Anwendungen, Vol. 12.) M. 24.00

- MOLLERUP (J.). Studier over den plane geometrie aksiomer. Kjöbenhavn, 1903. 8vo. 94 pp. K. 3.50
- MUZIO (E.). Trasformazione piana doppia del terzo ordine. Livorno, Giusti, 1903. 8vo. 24 pp.
- ONDRACEK (J.). Analytische Geometrie ebener Kurven in Büschel-Koordinaten. Heft I: Ebene Kurven in Normalen-Koordinaten erster Art. Wien, Gerold, 1903. 8vo. 5 + 32 pp. M. 1.20
- PECHE (M.). Welche Minimalflächen sind Träger einer Schaar reeller Hyperbeln? (Progr.) Breslau, 1903. 8vo. 16 pp.
- RICCI (G.). Origini e sviluppo dei moderni concetti fondamentali sulla geometria. (Annuario della r. Università degli studi di Padova per l'anno 1901-1902. Padova, 1902. 4to. 8 + 360 pp.)
- ROUCHÉ (E.) et LÉVY (L.). Analyse infinitésimale à l'usage des ingénieurs. Vol. II: Calcul intégral (intégrales indéfinies et définies; séries de Fourier; fonctions elliptiques; équations différentielles ordinaires et aux dérivées partielles; calcul des variations). Paris, Gauthier-Villars, 1902. 8vo. 11 + 848 pp. Fr. 15.00
- RUSSELL (B.). The principles of mathematics. Cambridge, University Press, 1903. 8vo. 29 + 534 pp. Cloth. 12s. 6d.
- SCHIEFFERS (G.). Bemerkungen zu einem Satze von Sophus Lie über algebraische Funktionen. 8vo. (*Berichte der mathematisch-physischen Klasse der K. Sächsischen Gesellschaft der Wissenschaften*, 1903, pp. 88-96.)
- SIMON (M.). Analytische Geometrie der Ebenen. 2te, verbesserte Auflage; 2ter Abdruck. Leipzig, Göschen, 1903. 12mo. 207 pp. Cloth. (Sammlung Göschen, No. 65.) M. 0.80
- STINER (G.). Ueber Durchschnittskurven von Flächen zweiten Grades; einige typische Formen der Kurven mit unpaaren Aesten. Winterthur, 1902. 4to. 16 pp., 6 plates. M. 1.80
- TANNERY (P.). See DESCARTES.
- TIKHOMANDRITSKY (M.). Differential and integral calculus, with exercises. Vol. II, part 1. St. Petersburg, 1903. 8vo. (Russian.) M. 7.50
- VERSLUYS (J.). Beknopte geschiedenis der wiskunde. Amsterdam, 1902. 8vo. 208 pp. M. 5.00
- WÖLFING (E.). Mathematischer Bücherschatz. Systematisches Verzeichnis der wichtigsten deutschen und ausländischen Lehrbücher und Monographien des 19. Jahrhunderts auf dem Gebiete der mathematischen Wissenschaften. In zwei Teilen. Teil I: Reine Mathematik. Mit einer Einleitung: Kritische Uebersicht über die bibliographischen Hilfsmittel der Mathematik. Leipzig, Teubner, 1903. 8vo. 36 + 416 pp. (Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, begründet von M. Cantor, Heft 16 I.) M. 14.00
- ZEEMANN (P.). Zuivere en toegepaste wiskunde. Delft, 1902. 8vo. 32 pp. M. 1.00

II. ELEMENTARY MATHEMATICS.

- BÖTTCHER (R.) und SENDLER (R.).** Raumlehre für Lehrerseminare. Nach dem Lehrplan vom 1. VII. 1901 bearbeitet. Teil I: Planimetrie. Breslau, Handel, 1903. 8vo. 4 + 98 pp. M. 1.00
- BOUVART (C.) et RATINET (A.).** Nouvelles tables de logarithmes à cinq décimales (division centésimale, conforme à l'arrêté ministériel du 3 août 1901), à l'usage des candidats aux écoles polytechnique et de Saint-Cyr. 2e édition, revue et corrigée. Paris, Hachette, 1903. 8vo. 128 pp. Fr. 2.00
- BRUHNS (C.).** Neues logarithmisch-trigonometrisches Handbuch auf sieben Decimalen. 6te Stereotyp-Ausgabe. Leipzig, Tauchnitz, 1903. 8vo. 24 + 610 pp. (Also English, French, and Italian editions.) M. 4.20
- BUTTEL (P.).** Raumlehre für Mittelschulen. Bearbeitet von A. Möller und H. Jarchov. Kiel, Lipsius & Fischer, 1903. 8vo. 9 + 208 pp. Cloth. M. 1.80
- COYM (G.).** Geometrie der Ebene. Teil I, erster Jahreskursus: Anschauungskursus der Geometrie und Elementarkursus der Konstruktionslehre. Leipzig, Schneider, 1903. 8vo. 67 pp. M. 0.80
- DIEKMANN (J.).** See KOPPE.
- GAUSS (F. G.).** Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 75te Auflage. Halle, 1903. 8vo. 166 + 35 pp. Half-morocco. M. 2.50
- . Kleine Ausgabe. Halle, 1903. 8vo. 4 + 97 pp. Cloth. M. 1.60
- GENAU (A.).** Geometrie für Lehrerbildungsanstalten. 11te Auflage, nach den Lehrplänen vom 1. VII. 1901 bearbeitet von A. Genau und J. Gründer. Büren, Hagen, 1903. 8vo. 3 + 255 pp. Cloth. M. 3.75
- GRÜNDER (J.).** See GENAU (A.).
- HALL (H. S.) and STEVENS (F. H.).** A school geometry. Part I: Lines and angles; rectilinear figures. Part II: Areas of rectilinear figures. (Containing the substance of Euclid, book I.) London and New York, Macmillan, 1903. 12mo. 10 + 140 pp. Cloth. \$0.40
- HARRISON (J.).** Practical plane and solid geometry for elementary students. Adapted to requirements of South Kensington Syllabus. London, Macmillan, 1903. 8vo. 264 pp. Cloth. 2s. 6d.
- HARST (A. D. VAN DER).** Over het invoeren van wortels door vermenigvuldiging bij goniometrische vergelijkingen. 's Gravenhage, 1902. 8vo. 28 pp. M. 0.50
- JARCHOV (H.).** See BUTTEL (P.).
- KNOPS (K.).** See KOPPE.
- KOPPE und Diekmann's Geometrie zum Gebrauch an höheren Unterrichtsanstalten.** Teil III. 2te Auflage. Ausgabe für Reallehranstalten. Grundlagen der darstellenden Geometrie; die wichtigsten Sätze über Kegelschnitte in elementar-synthetischer Behandlung. Von K. Knops. Analytische Geometrie der Ebene. Von J. Diekmann. Essen, Baedeker, 1903. 8vo. 6 + 240 pp. Cloth. M. 3.20

- KOPPESCHAAER (W. F.). Theorie der logarithmen, elementair behandeld. Culemborg, 1902. 8vo. 28 pp. M. 0.50
- . Theorie der machten en wortels, elementair behandeld. Culemborg, 1902. 8vo. 60 pp. M. 1.00
- KREUSCHMER. See LACKEMANN (C.).
- LACKEMANN (C.). Die Elemente der Geometrie. Ein Lehr- und Uebungsbuch für den geometrischen Unterricht an 6-klassischen höheren Lehranstalten. Teil 2: Trigonometrie und Stereometrie. 4te, verbesserte und vermehrte Auflage, nebst einem Anhang über die ersten Anfänge des Feldmessens. Unter Berücksichtigung der preussischen Lehrpläne vom Jahre 1901 bearbeitet von Kreuschmer. Breslau, Hirt, 1903. 8vo. 68 + 24 pp. Boards. M. 1.00
- M'AULAY (A.). Five figure logarithmic and other tables. London, Macmillan, 1903. 18mo. Cloth. 2s. 6d.
- MAYER (J. E.). Das mathematische Pensum des Primaners; ein Hilfsbuch für den Primaner humanistischer und realistischer Gymnasien. Heft 3 und 4: Stereometrie, Schluss; stereometrische Aufgaben mit vollständigen Auflösungen. Freiburg i. B., Lorenz, 1903. 8vo. M. 2.00
- MÖLLER (A.). See BUTTEL (P.).
- MÜLLER (C. H.) und PRESLER (O.). Leitfaden der Projektionslehre. Ein Uebungsbuch der konstruierenden Stereometrie. Ausgabe A: Vorzugsweise für Realgymnasien und Oberrealschulen. Leipzig, Teubner, 1903. 8vo. 8 + 320 pp. Cloth. M. 4.00
- . Ausgabe B: Für humanistische Gymnasien und sechsstufige Realanstalten. Leipzig, Teubner, 1903. 8vo. 6 + 138 pp. Cloth. M. 2.00
- MUSMACHER (C.). Leitfaden und Aufgabensammlung für den propädeutischen geometrischen Unterricht. Leipzig, Renger, 1903. 8vo. 32 pp. Boards. M. 0.50
- PIGEON (F.). Algèbre. Solutions des équations numériques de tous degrés par un nouveau système général pratique. 1e édition. Notre-Dame-d'Oë, Pigeon, 1903. 16mo. 91 pp. Fr. 1.50
- PINCHERLE (S.). Geometria pura elementare. 6ta edizione, con l'aggiunta delle figure sferiche. Milano, Hoepli, 1903. 16mo. 6 + 175 pp. (Manuali Hoepli.)
- PONS Y MERI (B.). Nociones de geometría. Valladolid, Montero, 1902. 8vo. 99 pp.
- PRESLER (O.). See MÜLLER (C. H.).
- RAJ-ABERRÓES. Nuevo descubrimiento matemático. La politomia 6 multisección geométrica del ángulo, del arco y de la circunferencia, y la consiguiente inscripción en ésta de cualquier polígono regular, sin tanteo ni empirismo, ni más adminículos que la regla y el compás. Cadiz, Ortiz, 1902. 8vo. 5 pp., 1 plate. Fr. 1.00
- RATINET (A.). See BOUVART (C.).
- RIBI (D.). Auflösungen zu den Aufgaben über die Elemente der Algebra. Revidiert von G. Wernly. Heft I: Zu Heft 1 und 2 der Aufgaben. 4te Auflage. Bern, Francke, 1903. 8vo. 63 pp. M. 1.10

SCHÜLKE (A.). Vierstellige Logarithmentafeln, nebst mathematischen, physikalischen und astronomischen Tabellen. Für den Schulgebrauch zusammengestellt. 4te verbesserte Auflage. Leipzig, Teubner, 1903. 8vo. M. 0.60

SENDLER (R.). See BÖTTCHER (R.).

SHMULEVICH (P.). Theoretical algebra. St. Petersburg, 1903. 8vo. 112 pp. (Russian.) M. 3.00

STEVENS (F. H.). See HALL (H. S.).

STBÖLL (A.). Elementi di geometria. Geometria intuitiva scritta pel Ilo, IIlo, IVO corso delle scuole reali e per istituti affini. 2a edizione, completamente rifatta. Con numerose questioni per esercizio. Wien, Hölder, 1903. 8vo. 3 + 222 pp. M. 3.20

TOLEDO (L. O. DE). Elementos de aritmetica universal. 2a edición. Parte I (calculatoria). Cuaderno 1. Madrid, 1903. 4to. Pp. 1-289. Fr. 8.50

WENTWORTH (G. A.). Plane and spherical trigonometry. 2d edition, revised. Boston, Ginn, 1903. 12mo. 7 + 307 + 25 pp. Half leather. \$1.35

WERNLY (G.). See RIBI (D.).

III. APPLIED MATHEMATICS.

ANDERHALDEN (B.). Die Aetherhypothesen von Descartes bis Fresnel, ihr Inhalt und ihre Entwicklung. Teil 1, 1901. Teil 2, 1902. Sarnen. 4to. 97 pp. M. 4.00

ATKINSON (A. A.). Electrical and magnetic calculations; for the use of electrical engineers and artisans, teachers, students, and all others interested in the theory and application of electricity and magnetism. 2d edition, revised. New York, Van Nostrand, 1903. 12mo. 7 + 310 pp. Cloth. \$1.50

AUERBACH (F.). See WINKELMANN (A.).

BORLETTI (F.). Formole e tavole per il calcolo delle risvolte al arco circolare, adattate alla divisione centesimale, ad uso degli ingegneri. Milano, 1902. 8vo. 12 + 68 pp. Fr. 3.00

BRYAN (G. H.). See ENCYKLOPÄDIE.

CAL (A. Y A.). Arquitectura naval; teoria de buque. Ferrol, 1902. 4to. 9 + 373 pp. Fr. 20.00

CANTOR (M.). Politische Arithmetik, oder die Arithmetik des täglichen Lebens. 2te Auflage. Leipzig, Teubner, 1903. 8vo. 10 + 156 pp. Cloth. M. 1.80

CAVALLI (E.). Avviamento allo studio della meccanica. Elementi di cinematica teorica. 2a edizione. Napoli, 1902. 8vo. 8 + 91 pp., 4 plates. Fr. 5.00

CHRISTEN (T.). Das Gesetz der Translation des Wassers in regelmässigen Kanälen, Flüssen und Röhren. Leipzig, Engelmann, 1903. 8vo. 7 + 169 pp., 2 plates. M. 5.00

COTTRELL (F. G.). Der Reststrom bei galvanischer Polarisation betrachtet als ein Diffusionsproblem. (Diss.) Leipzig, 1903. 8vo. 52 pp.

CRANZ (C.). See ENCYKLOPÄDIE.

DESSAU (B.). See RIGLI (A.).

DOLGORUKOV (R.). Theory of the motion of the moon. St. Petersburg, 1903. 8vo. (Russian.) M. 4.50

DOPPLER (C.). Ueber das farbige Licht der Doppelsterne und einiger anderer Gestirne des Himmels. Versuch einer das Bradley'sche Aberrations-Theorem als integrierenden Theil in sich schliessenden allgemeinen Theorie. Zur Feier seines 100. Geburtstages als erste Veröffentlichung des nach ihm benannten physikalischen Princips neu herausgegeben von F. J. Studnička. Prag, Rivnáč, 1903. 8vo. 25 pp., 1 plate, 1 portrait. M. 0.80

ENCYKLOPÄDIE der mathematischen Wissenschaften, mit Einschluss ihrer Anwendungen. Vol. IV: Mechanik, herausgegeben von F. Klein. Teil 2, Heft 2: Aërodynamik, von S. Finsterwalder; Ballistik, von C. Cranz. Leipzig, Teubner, 1903. 8vo. Pp. 149-279.

— Vol. V: Physik, herausgegeben von A. Sommerfeld. Teil 1, Heft 1: Maass und Messen, von C. Runge; Gravitation, von J. Zenneck; Allgemeine Grundlegung der Thermodynamik, von G. H. Bryan. Leipzig, Teubner, 1903. 8vo. Pp. 1-160.

FINSTERWALDER (S.). See ENCYKLOPÄDIE.

FLOTOW (A. von). Definitive Bahnbestimmung des Kometen 1863 I. (Diss.) Leipzig, 1902. 8vo. 21 pp.

GEDICUS (F. W.). Das System der Kinetik im Grundriss. Wiesbaden, Bergmann, 1903. 8vo. 8 + 78 pp. M. 1.80

GLOVER (J. W.). A graduation of the American experience table of mortality to Makeham's formula by the method of moments. 8vo. (*Transactions of the Actuarial Society of America*, Vol. 7, pp. 339-353.)

GRAETZ (L.). See WINKELMANN (A.).

HELMHOLTZ (H. von). Vorlesungen über theoretische Physik. Herausgegeben von A. König, O. Krüger-Menzel, F. Richarz, C. Runge. Vol. VI: Vorlesungen über Theorie der Wärme. Herausgegeben von F. Richarz. Leipzig, Barth, 1903. 8vo. 12 + 419 pp. Cloth. M. 16.00

IHERING (A. von). Die Gebläse; Bau und Berechnung der Maschinen zur Bewegung, Verdichtung und Verdünnung der Luft. 2te, umgearbeitete und vermehrte Auflage. Berlin, 1903. 8vo. 12 + 752 pp., 11 plates. Cloth. M. 20.00

KLEIN (F.). See ENCYKLOPÄDIE.

MALZI (J.). Ueber ein Gesetz der elektrolytischen Reaktion. (Diss.) Heidelberg, 1903. 8vo. 40 pp.

MANNO (R.). Theorie der Bewegungsübertragung als Versuch einer neuen Grundlegung der Mechanik. Leipzig, Engelmann, 1903. 8vo. 5 + 102 pp. M. 2.40

MÜLLER (R.). Leitfaden für die Vorlesungen über darstellende Geometrie an der herzoglichen technischen Hochschule zu Braunschweig. 2te Auflage. Braunschweig, Vieweg, 1903. 8vo. 8 + 95 pp. M. 2.50

- RICHARZ (F.). See HELMHOLTZ (H. VON).
- RIGHI (A.) und DESSAU (B.). Die Telegraphie ohne Draht. Braunschweig, Vieweg, 1903. 8vo. 11 + 481 pp. M. 12.00
- RUNGE (C.). See ENCYKLOPÄDIE.
- SCHENK (J.). Die Festigkeitsberechnung grösserer Drehstrommaschinen. Leipzig, Teubner, 1903. 8vo. 4 + 59 pp., 1 plate. M. 1.60
- SCHROEDER (C.). Die Rechenapparate der Gegenwart, gesammelt, geordnet, beschrieben und begutachtet. Magdeburg, 1903. 8vo. 112 pp. M. 2.00
- SOMMERFELD (A.). See ENCYKLOPÄDIE.
- STEWART (R. W.). Higher text-book of heat. London, 1903. 8vo. 396 pp. Cloth. 6s. 6d.
- STUDNICKA (F. J.). See DOPPLER (C.).
- VOLLER (A.). Grundlagen und Methoden der elektrischen Wellentelegraphie (sogenannten drahtlosen Telegraphie). Vortrag. Erweiterter Abdruck. Hamburg, Voss, 1903. 8vo. 52 pp. M. 1.80
- WINKELMANN (A.). Handbuch der Physik. Herausgegeben unter Mitwirkung von R. Abegg, P. Drude, K. Exner, H. Kayser und Anderen. 2te Auflage. (In 6 Bänden.) Vol. IV, 1ste Hälfte: Elektrizität und Magnetismus I. Bearbeitet von L. Graetz und F. Auerbach. Leipzig, Barth, 1903. 8vo. 6 + 384 pp. M. 12.00
- ZENNECK (J.). See ENCYKLOPÄDIE.

TWELFTH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND
SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES
TO THE PLACES OF THEIR PUBLICATION.

- ALLARDICE, R. E. On the Envelope of the Axes of a System of Conics Passing Through Three Fixed Points. Read (Chicago) Jan. 2, 1903. *Transactions of the American Mathematical Society*, vol. 4, No. 1, pp. 103-106; Jan., 1903.
- BLICHFELDT, H. F. On the Determination of the Distance between Two Points in Space of n Dimensions. Read (San Francisco) May 3, 1902. *Transactions of the American Mathematical Society*, vol. 3, No. 4, pp. 467-481; Oct., 1902.
- Note on a Property of the Conic Sections. Read (San Francisco) Dec. 20, 1902. *Bulletin of the American Mathematical Society*, vol. 9, No. 6, pp. 306-307; Mar., 1903.
- BLISS, G. A. The Geodesic Lines on the Anchor Ring. Read (Chicago) Dec. 27, 1900. *Annals of Mathematics*, 2d ser., vol. 4, No. 1, pp. 1-21; Oct., 1902.
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The following dates have been fixed for the meetings of the Society :

A. M.	A. M.
Sat., Oct. 31, 1903, 11:00	Sat., Feb. 27, 1904, 11:00
M.-Tue., Dec. 28-29, " "	" April 24, " "

The Tenth Summer Meeting and Fourth Colloquium of the Society will be held at the Massachusetts Institute of Technology, Boston, Mass., during the week beginning August 31, 1903.

